The spinorial geometry of supersymmetric backgrounds

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Abstract

We propose a new method to solve the Killing spinor equations of eleven-dimensional supergravity based on a description of spinors in terms of forms and on the $Spin(1,10)$ gauge symmetry of the supercovariant derivative. We give the canonical form of Killing spinors for backgrounds preserving two supersymmetries, $N=2$, provided that one of the spinors represents the orbit of $Spin(1,10)$ with stability subgroup $SU(5)$. We directly solve the Killing spinor equations of $N=1$ and some $N=2$, $N=3$ and $N=4$ backgrounds. In the $N=2$ case, we investigate backgrounds with $SU(5)$ and $SU(4)$ invariant Killing spinors and compute the associated spacetime forms. We find that $N=2$ backgrounds with $SU(5)$ invariant Killing spinors admit a timelike Killing vector and that the space transverse to the orbits of this vector field is a Hermitian manifold with an $SU(5)$-structure. Furthermore, $N=2$ backgrounds with $SU(4)$ invariant Killing spinors admit two Killing vectors, one timelike and one spacelike. The space transverse to the orbits of the former is an almost Hermitian manifold with an $SU(4)$-structure. The spacelike Killing vector field leaves the almost complex structure invariant. We explore the canonical form of Killing spinors for backgrounds preserving more than two supersymmetries, $N>2$. We investigate a class of $N=3$ and $N=4$ backgrounds with $SU(4)$ invariant spinors. We find that in both cases the space transverse to a timelike vector field is a Hermitian manifold equipped with an $SU(4)$-structure and admits two holomorphic Killing vector fields. We also present an application to M-theory Calabi-Yau compactifications with fluxes to one-dimension.

1 Introduction

The last ten years, there has been much activity in understanding the supersymmetric solutions of ten- and eleven-dimensional supergravities. This is because of the insight that these solutions give in string theory, M-theory and gauge theories, see e.g. [1, 2] and more recently [3, 4, 5]. Despite these developments, the supersymmetric solutions of eleven- and ten-dimensional supergravities have not been classified. This is mainly due to the fact that the supercovariant connections of supergravity theories in the presence of fluxes are not induced from connections of the tangent bundle of spacetime. However, progress has been made in two 'extreme' cases. On one end, J. Figueroa-O'Farrill and one of the authors classified the maximally supersymmetric solutions in eleven- and ten-dimensional supergravities [6, 7]. On the other end, J. Gauntlett, J. Gutowski and S. Pakis have solved the Killing spinor equations of eleven-dimensional supergravity in the presence of one Killing spinor [8, 9].

The main aim of this paper is to propose a new method to solve the Killing spinor equations of eleven-dimensional supergravity for any number of Killing spinors. Our method is based on the systematic understanding of spinors that can occur as solutions to the eleven-dimensional supergravity Killing spinor equations and the observation that the manifest gauge symmetry of the eleven-dimensional supercovariant derivative is $Spin(1,10)$. Because of this, as we shall explain, the Killing spinors can be put into a canonical form using $Spin(1,10)$ gauge transformations. Another ingredient of the method is the understanding of the stability subgroups in $Spin(1,10)$ of any number of spinors. For this in the first part of this paper, we present a description of spinors for an eleven-dimensional spacetime in terms of forms by adapting a formalism developed in [10] in the context of special holonomy. This description simplifies the task of classifying supersymmetric backgrounds in two ways.

- First it introduces a basis in the space of spinors which can be used to *directly solve* the Killing spinor equations.
- Second, it provides a systematic way to find the stability subgroup in $Spin(1,10)$ of N spinors and to compute the spacetime forms associated with a pair of spinors.

The stability subgroup of Killing spinors in $Spin(1,10)$ is a way to characterize (classify) the Killing spinors for any number of supersymmetries. However, as we shall see, it is possible that different number of spinors can have the same stability subgroup. It is well known that there are two types of orbits, $\mathcal{O}_{SU(5)}$ and $\mathcal{O}_{Spin(7)}$, of $Spin(1,10)$ in the space of Majorana spinors, Δ_{32}, with stability subgroups $SU(5)$ and $(Spin(7)\ltimes\mathbb{R}^8)\times\mathbb{R}$, respectively [11, 12], see also [13]. We give representatives for these two orbits in our formalism. We then use them to compute the associated spacetime forms. We shall see that the relations between the spacetime forms are manifest and there is no need to use Fierz identities. We would like to point out that another basis in the space of spinors has also been used in [14] to directly solve the Killing spinor equations, for some N, of a seven-dimensional supergravity[1]. This formalism did not employ the description

[1]We thank O. Mac Conamhna for explaining this result to us.

of spinors in terms of forms that we use. It would be instructive to compare the two methods for the same supergravity.

Next, we find the stability subgroups and the representatives of orbits for more than one spinor. We find that the stability subgroup of two generic spinors, if one of them has stability subgroup $SU(5)$, is the identity $\{1\}$. This has also been observed for two spinors in seven-dimensions [14]. However there are several special choices of two spinors with stability subgroups for example $SU(5)$, $SU(4)$, $SU(2)\times SU(3)$, $SU(2)\times SU(2)$, $Sp(2)$, $SU(3)$ and $SU(2)$. Using the $Spin(1,10)$ gauge symmetry of the supercovariant connection to put the Killing spinors into a canonical form, we give the most general expression for the second Killing spinor provided that the first one represents the orbit $\mathcal{O}_{SU(5)}$. We also compute the spacetime forms associated with the spinors with stability subgroups $SU(5)$ and $SU(4)$. We shall use these forms to provide a geometric characterization of the associated supersymmetric background.

Using the basis in the space of spinors that we have mentioned, we reduce the Killing spinor equations of eleven-dimensional supergravity to a number of differential and algebraic conditions which do not contain products of gamma matrices. To demonstrate the effectiveness of our formalism, we *directly solve* the Killing spinor equations of eleven-dimensional supergravity for backgrounds preserving one supersymmetry, $N=1$ backgrounds[2], provided that the Killing spinor represents the orbit $\mathcal{O}_{SU(5)}$ of $Spin(1,10)$ in Δ_{32}. The fluxes are explicitly related to the spacetime geometry. The spacetime admits a timelike Killing vector field and the space transverse to the orbits of this vector field is an almost Hermitian manifold. This is in agreement with the results of [8] which have been derived using a different method.

Next, we focus on the Killing spinor equations for backgrounds with $N=2$ supersymmetry. We solve the Killing spinor equation for the *most general* $N=2$ background that admits $SU(5)$ invariant Killing spinors. In particular, we express the fluxes in terms of the geometry of the spacetime. We find that the spacetime admits a time-like Killing vector and that the manifold transverse to the orbits of the Killing vector is Hermitian with an $SU(5)$-structure. We also solve the Killing spinor equations for a class of $N=2$ backgrounds that admit $SU(4)$ invariant spinors. We find that the spacetime admits a timelike Killing vector field and a spacelike vector field. The space transverse to the former is an almost Hermitian manifold with an $SU(4)$-structure which we determine. The almost complex structure is invariant under the action of the spacelike vector field. It is worth pointing out that if the Killing spinors are invariant under a subgroup $G\subset Spin(10,1)$, then the spacetime admits a geometric G-structure[3].

We also explain how our formalism can be used to classify $N>2$ supersymmetric backgrounds. We find that there are many classes of backgrounds with $N>2$ supersymmetry for which the spinors have different stability subgroups. In particular we present an example of such backgrounds for which the Killing spinors have stability subgroups $SU(n)$, $n\leq 5$, which we call the SU series. This class of backgrounds can be used to investigate M-theory Calabi-Yau compactifications with fluxes to one, three, five and

[2]From now on, N is the number of Killing spinors of a supersymmetric background.

[3]Note however that one can choose spinors which have stability subgroup $\{1\}$. This may limit the applicability of the G-structure approach for solving the Killing spinor equations because in such a case any form on the spacetime is invariant.

seven dimensions.

We investigate two classes of backgrounds with $N = 3$ and $N = 4$ supersymmetry which admit $SU(4)$ invariant Killing spinors. In both cases, we solve the Killing spinor equations and express the fluxes in terms of the geometry of spacetime. We find that the spacetime admits one timelike and two spacelike Killing vector fields. The space B transverse to the former is a Hermitian manifold equipped with an $SU(4)$-structure. The two spacelike Killing vectors are holomorphic on B. The space $\hat{B}$ transverse to all three Killing vectors is again a Hermitian manifold with an $SU(4)$-structure.

As an application, we use our results for $N = 1$ and $N = 2$ backgrounds with $SU(5)$ invariant Killing spinors to explore M-theory Calabi-Yau compactifications with fluxes to one-dimension. We define Calabi-Yau compactifications with fluxes to one-dimension to be on backgrounds which are invariant under the Poincaré group of one-dimensional Minkowski space and admit $SU(5)$ invariant Killing spinors[4]. We find that such backgrounds can have one or two supersymmetries. We derive the conditions on the spacetime geometry in both cases. In the latter case the manifold is a product of the real line and a ten-dimensional Calabi-Yau manifold. The non-trivial part of the fluxes is given by a traceless closed (2,2)-form on the Calabi-Yau manifold.

To illustrate the general method, the supercovariant connection of eleven-dimensional supergravity [15] is

$$\mathcal{D}_M = \nabla_M + \Sigma_M \ , \tag{1.1}$$

where

$$\nabla_M = \partial_M + \frac{1}{4}\Omega_{M,AB}\Gamma^{AB} \tag{1.2}$$

is the spin covariant derivative induced from the Levi-Civita connection,

$$\Sigma_M = -\frac{1}{288}(\Gamma_M{}^{PQRS}\mathrm{F}_{PQRS} - 8\mathrm{F}_{MPQR}\Gamma^{PQR}) \ , \tag{1.3}$$

F is the four-form field strength (or flux) and $M, N, P, Q, R, S = 0, \dots, 9, 10$ are spacetime indices. To find the Killing spinors is equivalent to solve the parallel transport problem for the supercovariant connection. This is related to the holonomy of the supercovariant connection. It is known that the holonomy of the supercovariant connection is $SL(32, \mathbb{R})$ [16, 17]. At this point, it is crucial to distinguish between the holonomy group $SL(32, \mathbb{R})$ and the gauge group $Spin(1, 10)$ of the supercovariant connection. The latter are the gauge transformations which leave the form of the supercovariant connection invariant and therefore are the manifest symmetries of the theory. Although $SL(32, \mathbb{R})$ is the holonomy of the supercovariant connection, the $SL(32, \mathbb{R})$ gauge transformations, $\mathcal{D}_M \to A^{-1}\mathcal{D}_M A$, mix the various terms in $\mathcal{D}_M$ that have different powers of gamma matrices. As a result, it acts non-trivially of the Levi-Civita connection and the 4-form field strength F. On the other hand the $Spin(1, 10)$ gauge transformations U give

$$\mathcal{D}_M(e, \mathrm{F}) \to U^{-1}\mathcal{D}_M U = \mathcal{D}_M(e', \mathrm{F}') \ , \tag{1.4}$$

where the frame e and the form field strength F are related to e' and F$'$ with a local Lorentz rotation.

[4]One may in addition require that the internal manifold is compact.

The existence of Killing spinors is characterized by the reduction of the holonomy group to subgroups of $SL(32,\mathbb{R})$ which have been given in [16, 17] and computed for many supersymmetric backgrounds in [18, 19]. For a background with N Killing spinors, local $SL(32,\mathbb{R})$ transformations can be used to bring them along the first N vectors in the standard basis of $\Delta_{32} = \mathbb{R}^{32}$ vector space. This is the 'standard' basis or canonical form for N Killing spinors up to local $SL(32,\mathbb{R})$ transformations. The simplicity of this result is facilitated by the property of $SL(32,\mathbb{R})$ to have one orbit in $\Delta_{32} - \{0\}$. However as we have explained such transformations will not leave the form of the supercovariant connection invariant. Because of this, it is preferable to find the canonical form of Killing spinors up to $Spin(1,10)$ gauge transformations. Because $Spin(1,10) \subset SL(32,\mathbb{R})$, $Spin(1,10)$ has more orbits in Δ_{32}. In addition, there more subgroups in $Spin(1,10)$ that preserve N spinors. As a consequence there are many more canonical forms for N spinors up to $Spin(1,10)$ gauge transformations.

Having found the canonical form for N spinors $\{\eta^I : I = 1,\dots N\}$ up to $Spin(1,10)$ gauge transformations, one substitutes them into the Killing spinor equations. The resulting equations can be rather involved. However as we have mentioned, there is a basis in the space of spinors which is natural within our formalism that can be used to reduce the Killing spinor equations to a set of differential and algebraic equations which do not involve gamma matrices. This method works in *all cases* and it can be used to directly solve the Killing spinor equations. In addition it becomes particularly simple and effective for Killing spinors that have a large stability subgroup in $Spin(1,10)$.

The paper has been organized as follows:

In section two, we give a description of the spinors, Δ_{32}, of eleven-dimensional supergravity in terms of forms and explain how to compute the spacetime forms associated with a pair of spinors. In section three, we give a representative of the orbit $\mathcal{O}_{SU(5)}$ in our formalism and compute the associated spacetime forms. We also present a basis in the space of spinors which we use later to analyze the Killing spinor equations. In section four, we give the canonical form of the two spinors that are associated with $N=2$ backgrounds provided that the first spinor is a representative of the orbit $\mathcal{O}_{SU(5)}$. We find that a generic pair of spinors have the identity $\{1\}$ as stability subgroup in $Spin(1,10)$. However there are several special examples with larger stability subgroups, like $SU(5)$, $SU(4)$ and others. We give explicitly the spacetime forms associated with the spinors with stability subgroups $SU(5)$ and $SU(4)$. In section five, we directly solve the Killing spinor equations for $N=1$ backgrounds with a Killing spinor which represents the orbit $\mathcal{O}_{SU(5)}$ of the $Spin(1,10)$ gauge group in Δ_{32}. We also investigate the geometry of the spacetime. In section six, we solve the Killing spinor equations for $N=2$ backgrounds for which the stability subgroup of the Killing spinors is $SU(5)$ and analyze the geometry of the underlying spacetime. In section seven, we solve the Killing spinor equations for $N=2$ backgrounds for which the stability subgroup of the Killing spinors is $SU(4)$ and analyze the geometry of the underlying spacetime. In section eight, we examine the Killing spinors that can occur in backgrounds with more than two supersymmetries. In sections nine and ten, we investigate the geometry of a class of $N=3$ and $N=4$ backgrounds, respectively. In section eleven, we apply our results to investigate M-theory Calabi-Yau compactifications with fluxes. In section twelve, we present our conclusions.

In appendix A, we investigate the orbits of $SU(5)$ and $SU(4)$ on the space of two

forms. These orbits are needed to understand the canonical form of Killing spinors for $N \geq 2$ backgrounds. In appendix B, we present some aspects of G-structures which we use to analyze the geometry of backgrounds preserving one and two supersymmetries. In appendix C, we give a representative of the orbit $\mathcal{O}_{Spin(7)}$. In appendix D, we present the solution of the Killing spinor equations for some $N = 2$ backgrounds with $SU(4)$ invariant Killing spinors.

2 Spinors from forms

To find the stability subgroups of spinors in the context of eleven-dimensional supergravity and to simplify many of the computations, we shall use a characterization of spinors in terms of forms. This has been explained for example in [20, 21] and has been used in [10] in the context of manifolds with special holonomy. We shall adapt the construction here for the spinors of eleven-dimensional supergravity. This method can be extended to spinors in all dimensions and all signatures.

A convenient way to describe the Majorana spin representation Δ_{32} of $Spin(1,10)$ is to begin from the spin representations of $Spin(10)$. Let $V = \mathbb{R}^{10}$ be a real vector space equipped with the standard Euclidean inner product. The complex spin (Dirac) representation of $Spin(10)$, Δ_c, is reducible and decomposes into two irreducible representations, $\Delta_c = \Delta^+_{16} \oplus \Delta^-_{16}$.

To construct these spin representations let $e_1, \dots, e_{10}$ be an orthonormal basis in $V = \mathbb{R}^{10}$ and J be a complex structure in V, $J(e_i) = e_{i+5}$, $i < 6$. Next consider the subspace $U = \mathbb{R}^5$ generated by $e_1, \dots, e_5$. Clearly $V = U \oplus J(U)$. The Euclidean inner product on V can be extended to a Hermitian inner product in $V_{\mathbb{C}} = V \otimes \mathbb{C}$ denoted by $<,>$, i.e.

$$< z^i e_i, w^j e_j >= \sum_{i=1}^{10} \bar{z}^i w^i \ , \tag{2.1}$$

where $\bar{z}^i$ is the standard complex conjugate of z^i in $V_{\mathbb{C}}$.

The space of Dirac spinors is $\Delta_c = \Lambda^*(U_{\mathbb{C}})$, where $U_{\mathbb{C}} = U \otimes \mathbb{C}$. The above inner product can be easily extended to Δ_c and it is called the Dirac inner product on the space of spinors. The gamma matrices act on Δ_c as

$$\begin{aligned} \Gamma_i \eta &= e_i \wedge \eta + e_i \lrcorner \eta \ , \qquad i \leq 5 \\ \Gamma_{5+i} \eta &= i e_i \wedge \eta - i e_i \lrcorner \eta \ , \qquad i \leq 5 \ , \end{aligned} \tag{2.2}$$

where $e_i \lrcorner$ is the adjoint of $e_i \wedge$ with respect to $<,>$. Moreover we have $\Delta^+_{16} = \Lambda^{\rm even} U_{\mathbb{C}}$ and $\Delta^-_{16} = \Lambda^{\rm odd} U_{\mathbb{C}}$. Clearly $\Gamma_i : \Delta^\pm_{16} \to \Delta^\mp_{16}$. The linear maps Γ_i are Hermitian with respect to the inner product $<,>$, $< \Gamma_i \eta, \theta >=< \eta, \Gamma_i \theta >$, and satisfy the Clifford algebra relations $\Gamma_i \Gamma_j + \Gamma_j \Gamma_i = 2\delta_{ij}$.

The charge conjugation matrix is constructed by first defining the map $B = \Gamma_6 \dots \Gamma_\natural$, where[5] $\Gamma_\natural = \Gamma_{10}$. Then the spinor inner product on Δ_c, which we denote with the same symbol, is

$$B(\eta, \theta) =< B(\bar{\eta}), \theta > \ , \tag{2.3}$$

[5] Form here on, we shall adopt the notation to denote the tenth direction with $\natural = 10$.

where $\bar{\eta}$ is the standard complex conjugate of η in $\Lambda^*(V_{\mathbb{C}})$. It is easy to verify that $B(\eta,\theta) = -B(\theta,\eta)$, i.e. B is skew-symmetric.

It remains to construct Γ_0 in this representation and impose the Majorana condition on the spinors. In this case, $\Gamma_0 = \pm\Gamma_1 \dots \Gamma_\natural$ (in what follows we shall choose the plus sign). It is easy to see that $\Gamma_0^2 = -1$ as expected and that Γ_0 anticommutes with Γ_i. The Majorana condition can be easily imposed by setting

$$\bar{\eta} = \Gamma_0 B(\eta) \ , \qquad \eta \in \Delta_c \ . \tag{2.4}$$

The Majorana spinors Δ_{32} of eleven-dimensional supergravity are those spinors in Δ_c which obey the Majorana condition (2.4). The Pin(10)-invariant inner product B induces a Spin(1,10) invariant inner product on Δ_{32} which is the usual skew-symmetric inner product on the space of spinors of eleven-dimensional supergravity. This completes the description of the spinors of eleven-dimensional supergravity, Δ_{32}, in terms of forms.

One advantage of describing spinor in terms of forms as above is that it allows us to find the stability subgroups in $Spin(1,10)$ which leave certain spinors invariant. It is also useful to bring spinors into a canonical or normal form using gauge transformations. In turn, these simplify the computation of the space-time forms

$$\alpha^{IJ} = \alpha(\eta^I,\eta^J) = \frac{1}{k!} B(\eta^I, \Gamma_{A_1\dots A_k}\eta^J) e^{A_1}\wedge\dots\wedge e^{A_k} \ , \quad I,J = 1,\dots,N \ , \qquad k = 0,\dots,9,\natural \tag{2.5}$$

associated with spinors. Note that it is sufficient to compute the forms up to degree five, the rest can be found using Poincare duality. Because of the symmetry properties of the gamma matrices and those of the B inner product, $\alpha^{IJ} = \alpha^{JI}$ for forms with degree $k = 1,2,5$ and $\alpha^{IJ} = -\alpha^{JI}$ for forms with degree $k = 0,3,4$. Therefore, it is sufficient to compute the spacetime forms with $I \leq J$.

3 $N = 1$

We shall begin with the investigation of the normal form of one spinor in eleven-dimensional supergravity. As we have mentioned there are two orbits $\mathcal{O}_{SU(5)}$ and $\mathcal{O}_{Spin(7)}$ of $Spin(1,10)$ in Δ_{32}, one with stability subgroup $SU(5)$ and the other with stability subgroup $(Spin(7) \ltimes \mathbb{R}^8) \times \mathbb{R}$. We shall mostly focus on the former case.

3.1 Spinors with stability group $SU(5)$

To find the normal form of a spinor up to an $SU(5) \subset Spin(10) \subset Spin(1,10)$ gauge transformation, we observe that Δ_c decomposes under $Spin(10)$ as

$$\Delta_c = \Delta_{16}^+ \oplus \Delta_{16}^- \ . \tag{3.1}$$

The representations $\Delta_{16}^\pm$ are complex. The Majorana condition selects a subspace Δ_{32} in Δ_c which intersects both $\Delta_{16}^\pm$. We have seen that $\Delta_{16}^\pm$ decompose under $SU(5)$ as

$$\Delta_{16}^+ \ = \ \sum_{k=0}^{2} \Lambda^{2k}(U_{\mathbb{C}}) \ ,$$

$$\Delta_{16}^{-} = \sum_{k=0}^{2} \Lambda^{2k+1}(U_{\mathbb{C}}) \ . \tag{3.2}$$

Clearly, the spinors that are invariant under $SU(5)$ are 1 and e_{12345}. Note that we have adopted the notation $e_1 \wedge \ldots \wedge e_k = e_{1\ldots k}$, e.g. $e_1 \wedge e_2 \wedge \ldots \wedge e_5 = e_{12345}$. Therefore the most general $SU(5)$-invariant spinor is

$$\eta = a1 + be_{12345} \ , \qquad a, b \in \mathbb{C} \ . \tag{3.3}$$

Imposing the Majorana condition on η, we find that $b = \bar{a}$. Therefore the $SU(5)$ invariant Majorana spinors are

$$\eta = a1 + \bar{a}e_{12345} \ . \tag{3.4}$$

So there are *two linearly independent real spinors* invariant under $SU(5)$ given by

$$\begin{aligned} \eta^{SU(5)} &= \frac{1}{\sqrt{2}}(1 + e_{12345}) \ , \\ \theta^{SU(5)} &= \frac{i}{\sqrt{2}}(1 - e_{12345}) \end{aligned} \tag{3.5}$$

which can represent the orbit $\mathcal{O}_{SU(5)}$. Indeed, these two spinors are in the same orbit of $Spin(1,10)$. To see this observe that

$$\theta^{SU(5)} = \Gamma_0 \eta^{SU(5)} = \Gamma_{1\ldots\natural} \eta^{SU(5)} \ . \tag{3.6}$$

Therefore the transformation in $Spin(1,10)$ which relates $\eta^{SU(5)}$ with $\theta^{SU(5)}$ projects onto the Lorentz element which is associated with reflection in all spatial directions. We shall see later that the forms associated with $\eta^{SU(5)}$ and $\theta^{SU(5)}$ are related by this Lorentz transformation. Therefore we conclude that in the $SU(5)$ case the parallel spinor η_1 can always be chosen as $\eta_1 = f\eta^{SU(5)}$, where f is a function of spacetime.

3.2 Spinors and antiholomorphic forms

It is convenient for many computations to use another basis in the space of spinors based on the isomorphism between spinors and $(0,p)$-forms[6]. In particular it is well known that

$$\Delta_c = \sum_{k=0}^{5} \Lambda^{0,k}(\mathbb{C}^5) \ . \tag{3.7}$$

To see this observe that the $SU(5)$ invariant spinor 1 satisfies

$$(\Gamma_j + i\Gamma_{j+5})1 = 0 \tag{3.8}$$

and similarly $(\Gamma_j - i\Gamma_{j+5})e_1 \wedge \ldots \wedge e_5 = 0$. This is the familiar property of the $SU(5)$ invariant spinors to be annihilated by the (anti)holomorphic gamma matrices. In particular, we define the gamma matrices in a Hermitian basis as

$$\Gamma_{\bar{\alpha}} = \frac{1}{\sqrt{2}}(\Gamma_\alpha + i\Gamma_{\alpha+5}) \ , \qquad \alpha = 1, \ldots, 5 \tag{3.9}$$

[6]On complex manifolds with an $Spin_c$ structure $\Delta_c = \Lambda^{0,*} \otimes \kappa^{\frac{1}{2}}$, where κ is the canonical bundle.

and $\Gamma^\alpha = g^{\alpha\bar\beta}\Gamma_\beta$, where $g_{\alpha\bar\beta} = \delta_{\alpha\bar\beta}$. The Clifford algebra relations in this basis are $\Gamma_\alpha\Gamma_{\bar\beta} + \Gamma_{\bar\beta}\Gamma_\alpha = 2g_{\alpha\bar\beta}$ and $\Gamma_\alpha\Gamma_\beta + \Gamma_\beta\Gamma_\alpha = \Gamma_{\bar\alpha}\Gamma_{\bar\beta} + \Gamma_{\bar\beta}\Gamma_{\bar\alpha} = 0$. The isomorphism (3.7) is simply

$$\Delta_c = \sum_{k=0}^{5} \Lambda^{0,k} \cdot 1 \ , \tag{3.10}$$

where $\cdot$ denotes Clifford multiplication. Therefore

$$\Gamma^{\bar\alpha_1\dots\bar\alpha_k} \cdot 1 \ , \qquad k = 0, \dots, 5 \tag{3.11}$$

is a *basis* in the space of spinors Δ_c. In particular, the other $SU(5)$ invariant spinor can be written as

$$e_{12345} = \frac{1}{8\cdot 5!}\epsilon_{\bar\alpha_1\dots\bar\alpha_5}\Gamma^{\bar\alpha_1\dots\bar\alpha_5} \cdot 1 \ , \tag{3.12}$$

where $\epsilon_{\bar1\bar2\bar3\bar4\bar5} = \sqrt{2}$. We shall extensively use this basis for spinors to analyze the Killing spinor equations.

3.2.1 Forms associated to $\eta^{SU(5)}$

The spacetime forms (2.5) associated to the spinor $\eta^{SU(5)}$ are easily computed. For this first observe that

$$\begin{aligned} B(1,1) &= B(e_{12345}, e_{12345}) = 0 \\ B(1, e_{12345}) &= -i \ . \end{aligned} \tag{3.13}$$

Using these we find that the non-vanishing spacetime forms are the following:

(i) A one-form

$$\kappa = \kappa(\eta^{SU(5)}, \eta^{SU(5)}) = B(\eta^{SU(5)}, \Gamma_0\eta^{SU(5)})e^0 = -e^0 \ , \tag{3.14}$$

(ii) a two-form

$$\omega = \omega(\eta^{SU(5)}, \eta^{SU(5)}) = -e^1\wedge e^6 - e^2\wedge e^7 - e^3\wedge e^8 - e^4\wedge e^9 - e^5\wedge e^\natural \tag{3.15}$$

and (iii) a five-form

$$\tau(\eta^{SU(5)}, \eta^{SU(5)}) = \mathrm{Im}[(e^1 + ie^6)\wedge\ldots\wedge(e^5 + ie^\natural)] + \frac{1}{2}e^0\wedge\omega\wedge\omega \,. \tag{3.16}$$

All these forms are $SU(5)$ invariant because the associated spinor is $SU(5)$ invariant. The presence of ω and the first part of τ may have been expected because of the $SU(5)$ invariance. There is also a time-like vector field κ. Having found the forms explicitly, it is straightforward to establish their relations, i.e. $i_\kappa\tau = \frac{1}{2}\omega\wedge\omega$. The forms κ, ω and τ and their relations have also been computed in [8] using different conventions and another method which involved Fierz identities.

4 $N = 2$

Let η_1 and η_2 be the Killing spinors of a background with $N = 2$ supersymmetry. It is always possible to choose η_1 up to an $Spin(1, 10)$ gauge transformation to be proportional either to $\eta^{SU(5)}$ or to $\eta^{Spin(7)}$ ($\eta^{Spin(7)}$ is given in appendix C). Suppose that $\eta_1 = f\eta^{SU(5)}$. One restriction on the choice of the second Killing spinor η_2 is that it must be linearly independent from η_1 at every spacetime point. This is because if two Killing spinors are linearly dependent at one spacetime point, since the Killing spinor equation is first order, they will be linearly dependent everywhere and so they will coincide (up to a constant overall scale). In addition, it is sufficient to determine η_2 up to $SU(5)$ gauge transformations that fix η_1. Using this gauge freedom, we decompose Δ^+_{16} under $SU(5)$. The second spinor can be chosen as any of the representatives of the orbits of $SU(5)$ in Δ^+_{16} which are linearly independent from the component 1 of η_1. This is sufficient because the Majorana condition determines the component of the spinor in Δ^-_{16}.

As we have already explained, the Δ^+_{16} representation of $Spin(10)$ decomposes under $SU(5)$ as

$$\Delta^+_{16} = \Lambda^0_1(\mathbb{C}^5) \oplus \Lambda^2_{10}(\mathbb{C}^5) \oplus \Lambda^4_{\bar{5}}(\mathbb{C}^5) \ . \tag{4.1}$$

In this notation the superscript denotes the degree of the forms and the subscript the dimension of the $SU(5)$ representation. Since $1 \in \Lambda^0_1(\mathbb{C}^5)$, it is clear that the second (Killing) spinor can be chosen as

$$\eta_2 = b1 + \theta + \text{c.c.} \ , \quad b \in \mathbb{C} \tag{4.2}$$

where $\theta \in \Lambda^2_{10}(\mathbb{C}^5) \oplus \Lambda^4_{\bar{5}}(\mathbb{C}^5)$ and c.c denotes the Majorana complex conjugation. In the analysis of the Killing spinor equations the parameter a in (4.2), as well as the other parameters which parameterize the orbits of $SU(5)$, are promoted to spacetime functions.

There are four possibilities to choose θ:

- $\theta = 0$
- $\theta \in \Lambda^4_{\bar{5}}(\mathbb{C}^5)$,
- $\theta \in \Lambda^2_{10}(\mathbb{C}^5)$ and
- the generic case where $\theta \in \Lambda^4_{\bar{5}}(\mathbb{C}^5) \oplus \Lambda^2_{10}(\mathbb{C}^5)$.

In some of these cases, there are more than one type of orbit of $SU(5)$. This makes the description of the choice of η_2 rather involved. However the analysis can be simplified somewhat by considering the stability subgroup in $SU(5)$ that leaves invariant both η_1 and η_2. Typically, the cases with 'large' stability subgroups are simpler to analyze because the associated Killing spinors depend on fewer spacetime functions. Simplifications can also occur by allowing certain components of the spinor to vanish.

4.1 $\eta_1 = \eta^{SU(5)}$ and $\theta = 0$

The Killing spinors η_1 and η_2 are spanned by the two $SU(5)$ invariant spinors $\eta^{SU(5)}$ and $\theta^{SU(5)}$. Since we can always choose $\eta_1 = a\eta^{SU(5)}$, $a \in \mathbb{R}$, the second Killing spinor is

$$\eta_2 = b_1\eta^{SU(5)} + b_2\theta^{SU(5)} \ , \qquad b_1, b_2 \in \mathbb{R} \ . \tag{4.3}$$

The stability subgroup of both η_1 and η_2 is $SU(5)$. Both spinors represent the orbits of $SU(5)$ acting with the trivial representation in $\Lambda^0(\mathbb{C}^5) = \mathbb{C}$. The constants a, b_1, b_2 in the context of Killing spinor equations are promoted to spacetime functions f, g_1, g_2, respectively. So generically, the Killing spinors in this case are determined by three real functions.

Some special cases arise by allowing one or more of a, b_1, b_2 to vanish. The first Killing spinor is chosen such that $\eta_1 \neq 0$ so $a \neq 0$. Thus we have as special cases $b_1 = 0$ or $b_2 = 0$. However b_2 must not vanish, because if $b_2 = 0$, η_1 and η_2 are linearly dependent and so they coincide. Thus the only special case is that for which $b_1 = 0$. In this case the Killing spinors are $\eta_1 = a\eta^{SU(5)}$ and $\eta_2 = b\theta^{SU(5)}$, $b_2 = b$.

4.1.1 Forms associated to $\theta^{SU(5)}$

To compute the spacetime forms associated with η_1 and η_2, it is sufficient to give the forms associated with $\eta^{SU(5)}$ and $\theta^{SU(5)}$. The forms associated with η_1 and η_2 can be computed by taking appropriate linear combinations of those of $\eta^{SU(5)}$ and $\theta^{SU(5)}$. The non-vanishing forms associated with $\theta^{SU(5)}$ are a one-form

$$\kappa(\theta^{SU(5)}, \theta^{SU(5)}) = -e^0 \ , \tag{4.4}$$

a two-form

$$\omega(\theta^{SU(5)}, \theta^{SU(5)}) = -e^1 \wedge e^6 - e^2 \wedge e^7 - e^3 \wedge e^8 - e^4 \wedge e^9 - e^5 \wedge e^\natural \tag{4.5}$$

and a five form

$$\begin{aligned} \tau(\theta^{SU(5)}, \theta^{SU(5)}) &= -\mathrm{Im}[(e^1 + ie^6) \wedge \ldots \wedge (e^5 + ie^\natural)] \\ &\quad + \frac{1}{2} e^0 \wedge \omega(\theta^{SU(5)}, \theta^{SU(5)}) \wedge \omega(\theta^{SU(5)}, \theta^{SU(5)}) \, . \end{aligned} \tag{4.6}$$

Comparing these with the forms associated with $\eta^{SU(5)}$, we observe that $\kappa(\eta^{SU(5)}, \eta^{SU(5)}) = \kappa = \kappa(\theta^{SU(5)}, \theta^{SU(5)})$ and $\omega(\eta^{SU(5)}, \eta^{SU(5)}) = \omega = \omega(\theta^{SU(5)}, \theta^{SU(5)})$ but $\tau(\eta^{SU(5)}, \eta^{SU(5)}) = \tau$ is linearly independent from $\tau(\theta^{SU(5)}, \theta^{SU(5)})$.

4.1.2 Forms associated to $\eta^{SU(5)}$ and $\theta^{SU(5)}$

There are also forms associated with the pair of spinors $(\eta^{SU(5)}, \theta^{SU(5)})$. In particular, we have that there is a zero-form

$$\alpha(\eta^{SU(5)}, \theta^{SU(5)}) = -1 \, , \tag{4.7}$$

a four-form

$$\zeta(\eta^{SU(5)}, \theta^{SU(5)}) = \frac{1}{2}\omega \wedge \omega \tag{4.8}$$

and a five-form

$$\tau(\eta^{SU(5)}, \theta^{SU(5)}) = \mathrm{Re}[(e^1 + ie^6) \wedge \ldots \wedge (e^5 + ie^\natural)] . \tag{4.9}$$

Therefore the inner product α of $\eta^{SU(5)}$ and $\theta^{SU(5)}$ is non-degenerate.

4.2 $\eta_1 = \eta^{SU(5)}$ and $\theta \in \Lambda^4_{\bar{5}}(\mathbb{C}^5)$

There is only one type of orbit of $SU(5)$ in $\Lambda^4_{\bar{5}}(\mathbb{C}^5)$ with stability subgroup $SU(4)$, $\mathcal{O}_{SU(4)}$. A representative can be chosen as

$$e_{1234} \ . \tag{4.10}$$

Therefore, after imposing the Majorana condition, we find two real representatives

$$\eta^{SU(4)} = \frac{1}{\sqrt{2}}(e_5 + e_{1234}) \ , \tag{4.11}$$

and

$$\theta^{SU(4)} = \frac{i}{\sqrt{2}}(e_5 - e_{1234}) \ . \tag{4.12}$$

Observe that

$$\begin{aligned} \eta^{SU(4)} &= \Gamma_5 \eta^{SU(5)} \qquad \theta^{SU(4)} = -\Gamma_5 \theta^{SU(5)} = \Gamma_\natural \eta^{SU(5)} \\ \theta^{SU(4)} &= -\Gamma_0 \eta^{SU(4)} \ . \end{aligned} \tag{4.13}$$

Therefore the $SU(5)$ and $SU(4)$ invariant spinors are related by an $Spin(1,10)$ transformation which projects onto the space reflection $e_1 \to -e_1$, $e_2 \to -e_2$, $e_3 \to -e_3$ and $e_4 \to -e_4$. In addition, $\eta^{SU(4)}$ and $\theta^{SU(4)}$ related by the $SU(5)$ transformation

$$e_1 \to e_1 \ , \quad e_2 \to ie_2 \ , \quad e_3 \to ie_3 \ , \quad e_4 \to ie_4 \ , \quad e_5 \to ie_5 \ . \tag{4.14}$$

Therefore they represent the same $SU(5)$ orbit. So in the construction of the second Killing spinor η_2, we can choose either $\eta^{SU(4)}$ or $\theta^{SU(4)}$. So the most general $SU(4)$ invariant Killing spinor η_2 is

$$\eta_2 = b_1 \eta^{SU(5)} + b_2 \theta^{SU(5)} + b_3 \eta^{SU(4)} \ , \qquad b_1, b_2, b_3 \in \mathbb{R} \ . \tag{4.15}$$

In this $N = 2$ case the Killing spinors depend on four spacetime functions – η_1 depends on one function and η_2 depends on three. We also take $b_3 \neq 0$ because otherwise η_2 will reduce to the $SU(5)$ invariant spinor (4.3). A special case for which the two spinors lie in different orbits is $\eta_1 = a\eta^{SU(5)}$ and $\eta_2 = b\eta^{SU(4)}$.

4.2.1 The spacetime forms of $\eta^{SU(4)}$

To compute the forms associated with η_1 and η_2 in (4.15), it is sufficient to compute the spacetime forms associated with $\eta^{SU(4)}$ and the forms associated with the pairs $(\eta^{SU(5)}, \eta^{SU(4)})$ and $(\theta^{SU(5)}, \eta^{SU(4)})$ of spinors. First let us consider the spacetime forms $\kappa(\eta^{SU(4)}, \eta^{SU(4)})$, $\omega(\eta^{SU(4)}, \eta^{SU(4)})$ and $\tau(\eta^{SU(4)}, \eta^{SU(4)})$ associated with $\eta^{SU(4)}$. Using (4.13) and the forms associated with $\eta^{SU(5)}$, we find a one-form

$$\kappa(\eta^{SU(4)}, \eta^{SU(4)}) = -e^0 \ , \tag{4.16}$$

a two-form

$$\omega(\eta^{SU(4)}, \eta^{SU(4)}) = e^1 \wedge e^6 + e^2 \wedge e^7 + e^3 \wedge e^8 + e^4 \wedge e^9 - e^5 \wedge e^\natural \ , \tag{4.17}$$

and a five-form

$$\begin{aligned} \tau(\eta^{SU(4)}, \eta^{SU(4)}) \ &= \ \mathrm{Im}[(e^1 + ie^6) \wedge \ldots \wedge (-e^5 + ie^\natural)] \\ &\quad + \frac{1}{2} e^0 \wedge \omega(\eta^{SU(4)}, \eta^{SU(4)}) \wedge \omega(\eta^{SU(4)}, \eta^{SU(4)}) \, . \end{aligned} \tag{4.18}$$

4.2.2 Forms associated to $(\eta^{SU(5)}, \eta^{SU(4)})$ and $(\theta^{SU(5)}, \eta^{SU(4)})$

Let us first consider the forms associated with the first pair $(\eta^{SU(5)}, \eta^{SU(4)})$. Using the relation (4.13) and the forms of $\eta^{SU(5)}$, one can find that there is a one-form

$$\kappa(\eta^{SU(5)}, \eta^{SU(4)}) = e^\natural \, , \tag{4.19}$$

a two-form

$$\omega(\eta^{SU(5)}, \eta^{SU(4)}) = -e^0 \wedge e^5 \ , \tag{4.20}$$

a three-form

$$\xi(\eta^{SU(5)}, \eta^{SU(4)}) = -\omega^{SU(4)} \wedge e^5 \ , \tag{4.21}$$

a four-form

$$\zeta(\eta^{SU(5)}, \eta^{SU(4)}) = \mathrm{Im}[(e^1 + ie^6) \wedge \ldots \wedge (e^4 + ie^9)] - e^0 \wedge \omega^{SU(4)} \wedge e^\natural \tag{4.22}$$

and a five form

$$\tau(\eta^{SU(5)}, \eta^{SU(4)}) = -e^0 \wedge \mathrm{Re}[(e^1 + ie^6) \wedge \ldots \wedge (e^4 + ie^9)] - \frac{1}{2} \omega^{SU(4)} \wedge \omega^{SU(4)} \wedge e^\natural \ , \tag{4.23}$$

where

$$\omega^{SU(4)} = e^1 \wedge e^6 + e^2 \wedge e^7 + e^3 \wedge e^8 + e^4 \wedge e^9 \ . \tag{4.24}$$

Observe that with these spinors there is an associated spacelike vector $\kappa(\eta^{SU(5)}, \eta^{SU(4)})$. Thus, although with a single spinor one can associate either a time-like or null vector field, with two or more spinors one can associate spacelike vectors as well.

The forms associated with the pair of spinors $(\theta^{SU(5)}, \eta^{SU(4)})$ can be calculated in a similar way to find a one-form

$$\kappa(\theta^{SU(5)}, \eta^{SU(4)}) = e^5 \ , \tag{4.25}$$

a two-form

$$\omega(\theta^{SU(5)}, \eta^{SU(4)}) = e^0 \wedge e^\natural \ , \tag{4.26}$$

a three-form

$$\xi(\theta^{SU(5)}, \eta^{SU(4)}) = \omega^{SU(4)} \wedge e^\natural \ , \tag{4.27}$$

a four-form

$$\zeta(\theta^{SU(5)}, \eta^{SU(4)}) = \mathrm{Re}[(e^1 + ie^6) \wedge \ldots \wedge (e^4 + ie^9)] - e^0 \wedge \omega^{SU(4)} \wedge e^5 \tag{4.28}$$

and a five-form

$$\tau(\theta^{SU(5)}, \eta^{SU(4)}) = e^0 \wedge \mathrm{Im}[(e^1 + ie^6) \wedge \ldots \wedge (e^4 + ie^9)] - \frac{1}{2}\omega^{SU(4)} \wedge \omega^{SU(4)} \wedge e^5 \ . \tag{4.29}$$

4.3 $\eta_1 = \eta^{SU(5)}$ and $\theta \in \Lambda^2_{10}(\mathbb{C}^5)$

There are three different orbits of $SU(5)$ in $\Lambda^2_{10}(\mathbb{C}^5)$ with different stability subgroups, see appendix A. The generic orbit is $\mathcal{O}_{SU(2)\times SU(2)}$, has stability subgroup $SU(2) \times SU(2)$ and a representative is

$$\lambda_1 e_{12} + \lambda_2 e_{34} \ , \qquad \lambda_1, \lambda_2 \in \mathbb{R} \ , \quad \lambda_1 \neq \lambda_2 \neq 0 \ . \tag{4.30}$$

The associated Majorana spinors are

$$\begin{aligned} \eta^{SU(2)\times SU(2)} &= \frac{1}{\sqrt{2}}(\lambda_1 e_{12} + \lambda_2 e_{34} - \lambda_1 e_{345} - \lambda_2 e_{125}) \ , \qquad \lambda_1^2 + \lambda_2^2 = 1 \\ \theta^{SU(2)\times SU(2)} &= \frac{i}{\sqrt{2}}(\lambda_1 e_{12} + \lambda_2 e_{34} + \lambda_1 e_{345} + \lambda_2 e_{125}) \ . \end{aligned} \tag{4.31}$$

In addition there are two special orbits with different stability subgroups. One special orbit is $\mathcal{O}_{Sp(2)}$ for $\lambda_1 = \pm\lambda_2$ and so it has the representative

$$e_{12} \pm e_{34} \ . \tag{4.32}$$

The choice of sign corresponds to different embeddings of $Sp(2)$ in $SU(5)$. In what follows we choose the representative with the plus sign. The associated Majorana spinors are

$$\begin{aligned} \eta^{Sp(2)} &= \frac{1}{2}(e_{12} + e_{34} - e_{345} - e_{125}) \ , \\ \theta^{Sp(2)} &= \frac{i}{2}(e_{12} + e_{34} + e_{345} + e_{125}) \ . \end{aligned} \tag{4.33}$$

Incidentally, in this case there is one more complex $Sp(2)$ invariant spinor which lies in $\Lambda^4_{10}(\mathbb{C}^5)$ and it is given by wedging (4.32) with itself. The associated Majorana spinors are

$$\begin{aligned} \zeta^{Sp(2)} &= \frac{1}{\sqrt{2}}(e_5 + e_{1234}) \\ \mu^{Sp(2)} &= \frac{i}{\sqrt{2}}(e_5 - e_{1234}) \ . \end{aligned} \tag{4.34}$$

The spinors (4.33) and (4.34) together with $\eta^{SU(5)}$ and $\theta^{SU(5)}$ are the six spinors in Δ_{32} invariant under $Sp(2)$. It is known that the $Sp(2)$ group is the holonomy group of eight-dimensional hyper-Kähler manifolds. Supergravity backgrounds with $Sp(2)$ holonomy group have been investigated before in the context of branes [25, 26].

The third special orbit, $\mathcal{O}_{SU(2)\times SU(3)}$, arises whenever $\lambda_1 = 0$ or $\lambda_2 = 0$ and has stability subgroup $SU(2) \times SU(3)$. In the latter case, a representative is

$$e_{12} \ . \tag{4.35}$$

The associated Majorana spinors are

$$\begin{aligned} \eta^{SU(2)\times SU(3)} &= \frac{1}{\sqrt{2}}(e_{12} - e_{345}) \ , \\ \theta^{SU(2)\times SU(3)} &= \frac{i}{\sqrt{2}}(e_{12} + e_{345}) \ . \end{aligned} \tag{4.36}$$

4.4 $\eta_1 = \eta^{SU(5)}$ and $\theta \in \Lambda^4_5(\mathbb{C}^5) \oplus \Lambda^2_{10}(\mathbb{C}^5)$

We assume that the group $SU(5)$ acts non-trivially on both subspaces of $\Lambda^1_5(\mathbb{C}^5)\oplus\Lambda^2_{10}(\mathbb{C}^5)$. If it does not, then this case reduces to cases investigated in the previous sections. To find the generic orbit of $SU(5)$ in $\Lambda^1_5(\mathbb{C}^5) \oplus \Lambda^2_{10}(\mathbb{C}^5)$, we choose a generic element in $\Lambda^2_{10}(\mathbb{C}^5)$ and use the $SU(5)$ transformations to bring the two-form in its normal form (4.30). As we have explained, this has stability subgroup $SU(2)\times SU(2)$. This subgroup can then be used to bring the four-form component of the generic element in $\Lambda^1_5(\mathbb{C}^5) \oplus \Lambda^2_{10}(\mathbb{C}^5)$ into a normal form. As a result, a representative of a generic orbit of $SU(5)$ in $\Lambda^1_5(\mathbb{C}^5)\oplus\Lambda^2_{10}(\mathbb{C}^5)$ is

$$c_1 e_{2345} + c_2 e_{1245} + c_3 e_{1234} + b(\lambda_1 e_{12} + \lambda_2 e_{34}) \ , \tag{4.37}$$

where $\lambda_1, \lambda_2 \in \mathbb{R}$, $c_1 \ , c_2 \ , c_3, b \in \mathbb{C}$, and $\lambda_1 \neq \lambda_2 \neq 0$ and $c_1 \neq c_2 \neq c_3 \neq 0$. To fix a redundancy in the parametrization, we should also set $\lambda_1^2 + \lambda_2^2 = 1$. However in what follows we shall not do so.

The stability subgroup of $\eta_1 = a\eta^{SU(5)}$ and (4.37) is $\{1\}$. Since the stability subgroup is the identity, this case has the least residual symmetry. The complex spinor (4.37) will give rise Majorana spinors which can be used as Killing spinors for $N = 2$ backgrounds.

Apart from the generic orbit with representative (4.37), there are various special orbits which have non-trivial stability subgroups. Since we are interested in the case where $\theta \in \Lambda^4_5(\mathbb{C}^5) \oplus \Lambda^2_{10}(\mathbb{C}^5)$, we shall assume that at least one of c_1, c_2, c_3 and at least one of λ_1, λ_2 do not vanish.

First suppose that $\lambda_1 \neq \lambda_2 \neq 0$. If either c_1 or c_2 vanishes, then the stability subgroup is $SU(2)$. If both $c_1 = c_2 = 0$, then the stability subgroup is $SU(2) \times SU(2)$. If $c_1, c_2 \neq 0$ and $c_3 = 0$, the stability subgroup is $\{1\}$.

On the other hand if $\lambda_1 = \lambda_2$ and either c_1 or c_2 vanishes, then the stability subgroup is $Sp(1)$. The stability subgroup remains the same if in addition $c_3 = 0$.

Next suppose that either $\lambda_1 = 0$ or $\lambda_2 = 0$, say $\lambda_2 = 0$. If $c_1, c_3 \neq 0$ and $c_2 = 0$, a representative can be chosen as in (4.37) with $c_2 = \lambda_2 = 0$ and the stability subgroup is $SU(2)$. If $c_1 = c_2 = 0$ and $c_3 \neq 0$, the stability subgroup is $SU(2) \times SU(2)$ and a

representative is given as in (4.37) with $c_1 = c_2 = \lambda_2 = 0$. If $c_2 = c_3 = 0$ and $c_1 \neq 0$, the stability subgroup is $SU(3)$ and a representative is given by (4.37) with $c_2 = c_3 = \lambda_2 = 0$.

4.5 The generic case

The most general choice of spinors for $N = 2$ backgrounds provided one of the spinors represents the $\mathcal{O}_{SU(5)}$ orbit is

$$\begin{aligned} \eta_1 &= a\eta^{SU(5)} \\ \eta_2 &= b_1 1 + c_1 e_{2345} + c_2 e_{1245} + c_3 e_{1234} + b_2(\lambda_1 e_{12} + \lambda_2 e_{34}) + \text{c.c} \ , \end{aligned} \tag{4.38}$$

where the parameters are as in (4.37) and $b_1 \in \mathbb{C}$ and $b_2 = b$. For the generic case the stability subgroup of η_1 and η_2 is $\{1\}$. However there are several cases which have larger stability subgroups. Some examples of $N = 2$ spinors with larger stability subgroups are listed in the table below. Some others have already been mentioned in the previous sections.

Conditions on parameters	Stability subgroups
$c_1 = c_2 = c_3 = \lambda_1 = \lambda_2 = 0\ ,\ b_1 \neq 0$	$SU(5)$
$c_2 = c_3 = \lambda_1 = \lambda_2 = 0\ ,\ b_1, c_1 \neq 0$	$SU(4)$
$c_1 = c_2 = c_3 = \lambda_2 = 0\ ,\ b_1, \lambda_1 \neq 0$	$SU(2) \times SU(3)$
$c_1 = c_2 = c_3 = \lambda_2 = 0\ ,\ b_1, \lambda_1, \lambda_2 \neq 0\ , \lambda_1 \neq \lambda_2$	$SU(2) \times SU(2)$
$c_1 = c_2 = c_3 = \lambda_2 = 0\ ,\ b_1, \lambda_1, \lambda_2 \neq 0\ , \lambda_1 = \lambda_2$	$Sp(2)$
$c_1 = c_2 = \lambda_2 = 0\ ,\ b_1, \lambda_1, \lambda_2, c_3 \neq 0\ , \lambda_1 \neq \lambda_2$	$SU(2) \times SU(2)$
$c_1 = c_2 = \lambda_2 = 0\ ,\ b_1, \lambda_1, \lambda_2, c_3 \neq 0\ , \lambda_1 = \lambda_2$	$Sp(2)$
$c_1 = c_2 = \lambda_2 = 0\ ,\ b_1, \lambda_1, c_3 \neq 0$	$SU(2) \times SU(2)$
$c_2 = c_3 = \lambda_2 = 0\ ,\ b_1, \lambda_1, c_1 \neq 0$	$SU(3)$
$c_1 = 0\ ,\ b_1, \lambda_1, \lambda_2, c_2 \neq 0$	$SU(2)$
$b_1, \lambda_1, \lambda_2, c_1, c_2 \neq 0$	$\{1\}$
Generic	$\{1\}$

(4.39)

Some simplification may occur in the form of the spinor η_2 using considerations similar to those we have employed in the $SU(4)$ invariant case to exclude the presence of both $\eta^{SU(4)}$ and $\theta^{SU(4)}$ in η_2.

5 $N = 1$ backgrounds

5.1 The Killing spinor equations

To solve the Killing spinor equations, it is convenient to introduce an orthonormal frame $\{e^A : A = 0, \ldots, \natural\}$ and to write the spacetime metric as

$$ds^2 = -(e^0)^2 + \sum_{i=1}^{\natural} (e^i)^2 \ . \tag{5.1}$$

In this frame, the four-form field strength F can be expanded in electric and magnetic parts as

$$\mathrm{F} = \frac{1}{3!} e^0 \wedge G_{ijk} e^i \wedge e^j \wedge e^k + \frac{1}{4!} F_{ijkl} e^i \wedge e^j \wedge e^k \wedge e^l \ . \tag{5.2}$$

The spin (Levi-Civita) connection has non-vanishing components

$$\Omega_{0,ij}\ , \quad \Omega_{0,0j}\ , \quad \Omega_{i,0j}\ , \quad \Omega_{i,jk}\ . \tag{5.3}$$

In terms of these, the Killing spinor equation decomposes as

$$\begin{aligned}
0 &= \partial_0 \eta + \frac{1}{4}\Omega_{0,ij}\Gamma^{ij}\eta - \frac{1}{2}\Omega_{0,0i}\Gamma_0\Gamma^i\eta - \frac{1}{288}(\Gamma_0\Gamma^{ijkl}F_{ijkl} - 8G_{ijk}\Gamma^{ijk})\eta\,, \\
0 &= \partial_i \eta + \frac{1}{4}\Omega_{i,jk}\Gamma^{jk}\eta - \frac{1}{2}\Omega_{i,0j}\Gamma_0\Gamma^j\eta - \frac{1}{288}(\Gamma_i{}^{jklm}F_{jklm} \\
&\quad +4\Gamma_0\Gamma_i{}^{jkl}G_{jkl} - 24\Gamma_0 G_{ijk}\Gamma^{jk} - 8F_{ijkl}\Gamma^{jkl})\eta\,.
\end{aligned} \tag{5.4}$$

The simplification that occurs for $N = 1$ backgrounds is that there is always an $Spin(1,10)$ gauge transformation to bring the Killing spinors into the form $f\eta^{SU(5)}$ or $f\eta^{Spin(7)}$, where f is a function which is restricted by the Killing spinor equations. We shall focus on the $f\eta^{SU(5)}$ Killing spinor. Substituting this into the Killing spinor equations, we have

$$\begin{aligned}
\partial_0 \log(f)\eta^{SU(5)} &+ \frac{1}{4}\Omega_{0,ij}\Gamma^{ij}\eta^{SU(5)} - \frac{1}{2}\Omega_{0,0i}\Gamma_0\Gamma^i\eta^{SU(5)} \\
&- \frac{1}{288}(\Gamma_0\Gamma^{ijkl}F_{ijkl} - 8G_{ijk}\Gamma^{ijk})\eta^{SU(5)} = 0\,, \\
\partial_i \log(f)\eta^{SU(5)} &+ \frac{1}{4}\Omega_{i,jk}\Gamma^{jk}\eta^{SU(5)} - \frac{1}{2}\Omega_{i,0j}\Gamma_0\Gamma^j\eta^{SU(5)} - \frac{1}{288}(\Gamma_i{}^{jklm}F_{jklm} \\
&+4\Gamma_0\Gamma_i{}^{jkl}G_{jkl} - 24\Gamma_0 G_{ijk}\Gamma^{jk} - 8F_{ijkl}\Gamma^{jkl})\eta^{SU(5)} = 0\,.
\end{aligned} \tag{5.5}$$

To analyze these equations, we first observe that they can be written entirely as ten-dimensional equations using $\Gamma_0 1 = i1$ and $\Gamma_0 e_{12345} = -ie_{12345}$. Next, it is convenient to introduce the Hermitian basis (3.11) in the space of spinors and use the fact that the spinors 1 and e_{12345} are annihilated by the holomorphic and antiholomorphic gamma matrices, respectively, i.e. $\Gamma^\alpha 1 = 0$ and $\Gamma^{\bar\alpha} e_{12345} = 0$. In addition we write the metric, and decompose the spin connection and the fluxes in terms of the Hermitian basis. For example the magnetic part of the flux F decomposes as $(4,0)+(0,4)$, $(3,1)+(1,3)$ and $(2,2)$ forms in the usual way and similarly the electric part of the flux G decomposes as $(3,0)+(0,3)$ and $(2,1)+(1,2)$ forms. In addition observe that the spinor e_{12345} can be expressed in terms of the Hermitian basis as

$$e_{12345} = \frac{1}{(\sqrt{2})^5}\Gamma^{\bar 1 \bar 2 \dots \bar 5} 1 \ . \tag{5.6}$$

In this way, we rewrite the Killing spinor equations as the vanishing of a spinor expressed in terms of the Hermitian basis (3.11). For such a spinor to vanish all the components in the Hermitian basis should vanish. In particular, the first Killing spinor equation in (5.5) gives

$$\partial_0 \log f + \frac{1}{2}\Omega_{0,\alpha\bar\beta}g^{\alpha\bar\beta} - \frac{i}{24}F_\alpha{}^\alpha{}_\beta{}^\beta = 0 \ , \tag{5.7}$$

$$i\Omega_{0,0\bar\alpha} + \frac{1}{3} G_{\bar\alpha\beta}{}^{\beta} + \frac{i}{72} F_{\beta_1\beta_2\beta_3\beta_4} \epsilon^{\beta_1\beta_2\beta_3\beta_4}{}_{\bar\alpha} = 0 \ , \tag{5.8}$$

$$\Omega_{0,\bar\alpha\bar\beta} - \frac{i}{6} F_{\bar\alpha\bar\beta\gamma}{}^{\gamma} - \frac{1}{18} G_{\gamma_1\gamma_2\gamma_3} \epsilon^{\gamma_1\gamma_2\gamma_3}{}_{\bar\alpha\bar\beta} = 0 \ . \tag{5.9}$$

There are three more conditions arising from the Killing spinor equations but they are related to the ones above with (standard) complex conjugation and therefore are not independent. The equation (5.7) and its complex conjugate imply that

$$\partial_0 \log f = 0 \tag{5.10}$$

and

$$\Omega_{0,\alpha\bar\beta} g^{\alpha\bar\beta} = \frac{i}{12} F_{\alpha}{}^{\alpha}{}_{\beta}{}^{\beta} \ . \tag{5.11}$$

Therefore f is does not depend on the time frame direction and the trace of the magnetic part of F is determined by the Levi-Civita connection of the spacetime.

The second Killing spinor equation is decomposed in two parts which involve the derivative of the spinor along the holomorphic and anti-holomorphic frame directions, respectively. Since the Killing spinor equation is real, it is sufficient to consider the Killing spinor equation with derivatives along the anti-holomorphic frame directions. The other part involving derivatives along the holomorphic frame directions is determined from the anti-holomorphic one by standard complex conjugation and therefore it does not give independent equations. In particular, we find

$$\partial_{\bar\alpha} \log f + \frac{1}{2} \Omega_{\bar\alpha,\beta\bar\gamma} g^{\beta\bar\gamma} + \frac{i}{12} G_{\bar\alpha\gamma}{}^{\gamma} - \frac{1}{72} \epsilon_{\bar\alpha}{}^{\beta_1\beta_2\beta_3\beta_4} F_{\beta_1\beta_2\beta_3\beta_4} = 0 \ , \tag{5.12}$$

$$\partial_{\bar\alpha} \log f - \frac{1}{2} \Omega_{\bar\alpha,\beta\bar\gamma} g^{\beta\bar\gamma} + \frac{i}{4} G_{\bar\alpha\gamma}{}^{\gamma} = 0 \ , \tag{5.13}$$

$$i\Omega_{\bar\alpha,0\bar\beta} + \frac{1}{6} F_{\bar\alpha\bar\beta\gamma}{}^{\gamma} - \frac{i}{18} \epsilon_{\bar\alpha\bar\beta}{}^{\gamma_1\gamma_2\gamma_3} G_{\gamma_1\gamma_2\gamma_3} = 0 \ , \tag{5.14}$$

$$i\Omega_{\bar\alpha,0\beta} + \frac{1}{12} g_{\bar\alpha\beta} F_{\gamma}{}^{\gamma}{}_{\delta}{}^{\delta} + \frac{1}{2} F_{\bar\alpha\beta\gamma}{}^{\gamma} = 0 \ , \tag{5.15}$$

$$\Omega_{\bar\alpha,\bar\beta\bar\gamma} + \frac{i}{6} G_{\bar\alpha\bar\beta\bar\gamma} - \frac{1}{12} \epsilon_{\bar\alpha\bar\beta\bar\gamma}{}^{\gamma_1\gamma_2} F_{\gamma_1\gamma_2\delta}{}^{\delta} - \frac{1}{12} F_{\bar\alpha\gamma_1\gamma_2\gamma_3} \epsilon^{\gamma_1\gamma_2\gamma_3}{}_{\bar\beta\bar\gamma} = 0 \ , \tag{5.16}$$

$$\Omega_{\bar\alpha,\beta\gamma} - \frac{i}{2} G_{\bar\alpha\beta\gamma} - \frac{i}{3} g_{\bar\alpha[\beta} G_{\gamma]\delta}{}^{\delta} - \frac{1}{36} F_{\bar\alpha\bar\gamma_1\bar\gamma_2\bar\gamma_3} \epsilon^{\bar\gamma_1\bar\gamma_2\bar\gamma_3}{}_{\beta\gamma} = 0 \ . \tag{5.17}$$

These equations can be viewed either as conditions on the fluxes F and G or conditions on the spacetime geometry as represented by the Levi-Civita connection Ω. It turns out that it is convenient to express the fluxes in terms of the spacetime geometry.

5.2 The solution to the Killing spinor equations

The above Killing spinor equations can be solved to express the fluxes in terms of the Levi-Civita connection of spacetime Ω. In particular, subtracting (5.13) from (5.12), we find that

$$\Omega_{\bar\alpha,\beta}{}^{\beta} - \frac{i}{6} G_{\bar\alpha\beta}{}^{\beta} - \frac{1}{72} \epsilon_{\bar\alpha}{}^{\beta_1\ldots\beta_4} F_{\beta_1\ldots\beta_4} = 0 \ . \tag{5.18}$$

This equation together with (5.8) give

$$F_{\beta_1\dots\beta_4} = \frac{1}{2}(-\Omega_{0,0\bar{\alpha}} + 2\Omega_{\bar{\alpha},\beta}{}^{\beta})\epsilon^{\bar{\alpha}}{}_{\beta_1\dots\beta_4} \tag{5.19}$$

and

$$G_{\bar{\alpha}\beta}{}^{\beta} = -2i\Omega_{\bar{\alpha},\beta}{}^{\beta} - 2i\Omega_{0,0\bar{\alpha}} \ . \tag{5.20}$$

Observe that consistency of (5.15) with its complex conjugate requires that

$$\Omega_{\bar{\alpha},0\beta} + \Omega_{\beta,0\bar{\alpha}} = 0 \ . \tag{5.21}$$

We shall see in the investigation of the geometry that there is a frame such that this condition can always be satisfied. Next we take the trace of (5.15) to find

$$F_{\alpha}{}^{\alpha}{}_{\beta}{}^{\beta} = 12i\Omega_{\bar{\alpha},0\beta}g^{\bar{\alpha}\beta} \ . \tag{5.22}$$

Consistency with (5.11) requires that $\Omega_{0,\beta\bar{\alpha}}g^{\beta\bar{\alpha}} + \Omega_{\bar{\alpha},0\beta}g^{\bar{\alpha}\beta} = 0$. We shall see later that this condition is again satisfied with an appropriate choice of frame. Substituting (5.22) back into (5.15), we have

$$F_{\beta\bar{\alpha}\gamma}{}^{\gamma} = 2i\Omega_{\bar{\alpha},0\beta} + 2ig_{\bar{\alpha}\beta}\Omega_{\bar{\gamma},0\delta}g^{\bar{\gamma}\delta} \ . \tag{5.23}$$

Taking the trace of the (5.17) and using (5.19) and (5.20), we get

$$\Omega_{\bar{\alpha},\beta\gamma}g^{\bar{\alpha}\beta} - \Omega_{\gamma,\beta}{}^{\beta} - \Omega_{0,0\gamma} = 0 \ . \tag{5.24}$$

This is a condition on the geometry of spacetime which we shall investigate later. Substituting (5.24) back into (5.17), we find that

$$G_{\bar{\alpha}\beta\gamma} = -2i\Omega_{\bar{\alpha},\beta\gamma} + 2ig_{\bar{\alpha}[\beta}\Omega_{0,0\gamma]} \ . \tag{5.25}$$

To investigate the equations (5.14) and (5.16), first observe that consistency of (5.8) with (5.15) requires that

$$\Omega_{0,\bar{\alpha}\bar{\beta}} = \Omega_{\bar{\alpha},0\bar{\beta}} \ . \tag{5.26}$$

This again can be satisfied with an appropriate choice of a frame on the spacetime. These equations can be easily solved to reveal

$$G_{\bar{\alpha}_1\bar{\alpha}_2\bar{\alpha}_3} = 6i\Omega_{[\bar{\alpha}_1,\bar{\alpha}_2\bar{\alpha}_3]} \tag{5.27}$$

and

$$F_{\bar{\alpha}\beta_1\beta_2\beta_3} = \frac{1}{2}[\Omega_{\bar{\alpha},\bar{\gamma}_1\bar{\gamma}_2}\epsilon^{\bar{\gamma}_1\bar{\gamma}_2}{}_{\beta_1\beta_2\beta_3} + 3\Omega_{\bar{\gamma}_1,\bar{\gamma}_2\bar{\gamma}_3}\epsilon^{\bar{\gamma}_1\bar{\gamma}_2\bar{\gamma}_3}{}_{[\beta_1\beta_2}g_{\beta_3]\bar{\alpha}} + 12i\Omega_{[\beta_1,0\beta_2}g_{\beta_3]\bar{\alpha}}] \ . \tag{5.28}$$

It remains to solve (5.12). For this substitute (5.18) back into (5.12) and compare it with (5.8) to find

$$2\partial_{\bar{\alpha}}\log f + \Omega_{0,0\bar{\alpha}} = 0 \ . \tag{5.29}$$

To summarize the solution of the Killing spinor equations, the electric part of the flux G is completely determined in terms of the geometry. In particular the (0,3) part, $G^{0,3}$,

is given in (5.27) and the (2,1) part, $G^{2,1}$, is given in (5.25). The rest of the components are determined by standard complex conjugation. Similarly, the (4,0) and (3,1) parts of the magnetic flux F, $F^{4,0}$ and $F^{3,1}$, are determined in terms of the geometry in (5.19) and (5.28), respectively. The $F^{0,4}$ and $F^{1,3}$ components are also determined from $F^{4,0}$, and $F^{3,1}$ by standard complex conjugation. In addition, the trace of $F^{2,2}$ is determined in terms of the geometry in (5.23). The Killing spinor equations do not determine the traceless part of $F^{2,2}$ in terms of the geometry and do not involve the traceless part $\Omega_{i,\beta\bar{\gamma}} - \frac{1}{5}\Omega_{i,\delta}{}^{\delta} g_{\beta\bar{\gamma}}$ of the connection Ω.

5.3 The geometry of spacetime

The one-form $\kappa^f = -f^2\kappa = f^2 e^0$ is associated with a Killing vector field. To see this, we have to verify the Killing vector equation $\nabla_A \kappa^f_B + \nabla_B \kappa^f_A = 0$. The $(A,B) = (0,0)$ component is automatically satisfied. The $(A,B) = (0,\bar{\alpha})$ component is satisfied provided

$$\partial_{\bar{\alpha}} f^2 e^0 + \Omega_{0,0\bar{\alpha}} f^2 e^0 = 0 \tag{5.30}$$

which is satisfied because of (5.29). Similarly, the Killing vector equations along $(A,B) = (\alpha,\beta)$ and $(A,B) = (\alpha,\bar{\beta})$ are also satisfied because of (5.21) and (5.26).

Since κ^f is a timelike Killing vector field, one can always choose coordinates such that $\kappa^f = \partial_t$ and the metric can be written as

$$ds^2 = -f^4(dt+\alpha)^2 + ds^2_{10} \tag{5.31}$$

where the metric ds^2_{10} on the ten-dimensional space transverse to the orbits of κ^f, f and the one-form α are independent of t. A natural choice of frame is $e^0 = f^2(dt+\alpha)$ with the rest of the components e^i to be a frame of ds^2_{10}, i.e.

$$ds^2 = -(e^0)^2 + \sum_{i=1}^{10} (e^i)^2 \ . \tag{5.32}$$

Since the frame e^i does not depend on the time coordinate t, the torsion free condition, $de^i + \Omega^i{}_j \wedge e^j = 0$, of the Levi-Civita connection of the spacetime for this frame requires that

$$\Omega_{i,0j} = \Omega_{0,ij} \ . \tag{5.33}$$

As a consequence the conditions (5.21) and (5.26) are satisfied.

The only remaining condition on the geometry of the spacetime is (5.24). This restricts the geometry of the ten-dimensional space B which is transverse to the orbits of the time-like Killing vector field κ^f. Because of (5.29), this can be rewritten as

$$\Omega_{\bar{\alpha},\beta\gamma} g^{\bar{\alpha}\beta} - \Omega_{\gamma,\beta}{}^{\beta} + 2\partial_\gamma f = 0 \ . \tag{5.34}$$

The space B is an almost Hermitian manifold equipped with an $SU(5)$ invariant (5,0)+(0,5) form τ. Therefore one can use the Gray-Hervella classification of almost Hermitian manifolds to describe the geometry. In this context, (5.34) can be written using appendix B as

$$(\mathrm{w}_3)_\gamma + (\bar{\mathrm{w}}_5)_\gamma + 2\partial_\gamma f = 0 \ . \tag{5.35}$$

Equivalently, one can use $(W_5)_i = \frac{1}{40}\epsilon^{j_1 \dots j_5}\nabla_{[i}\epsilon_{j_1\dots j_5]}$ to write[7]

$$W_5 + 2df = 0 \ , \tag{5.36}$$

i.e. W_5 must be exact. This is the only condition on the geometry of the almost Hermitian ten-dimensional space B arising from the Killing spinor equations. The conditions on geometry of the spacetime that we have described for $N = 1$ are in agreement with those of [8] which have been derived using a different method. Of course there are more conditions on the geometry arising from the closure of the fluxes (the Bianchi identity) and the supergravity field equations.

6 $N = 2$ backgrounds with $SU(5)$ invariant Killing spinors

6.1 The Killing spinor equations

As we have explained the most general $SU(5)$ invariant Killing spinors are

$$\begin{aligned} \eta_1 &= f(1+e_{12345}) \ , \\ \eta_2 &= g_1(1+e_{12345}) + ig_2(1-e_{12345}) \ , \end{aligned} \tag{6.1}$$

where f, g_1 and g_2 are real functions on the spacetime which are restricted by the Killing spinor equations. We shall assume that $g_2 \neq 0$ because otherwise the second spinor will be linearly depend to the first one.

The Killing spinor equation for η_1 can be analyzed as in the $N = 1$ case that we have already explained. In particular observe that we can write

$$\mathcal{D}_M\eta_1 = \mathcal{D}_M[f(1+e_{12345})] = \partial_M f(1+e_{12345}) + f\mathcal{D}_M(1+e_{12345}) = 0 \tag{6.2}$$

and thus

$$\mathcal{D}_M(1+e_{12345}) = -\partial_M \log f(1+e_{12345}) \ . \tag{6.3}$$

Using this, we can rewrite the Killing spinor equation of η_2 as

$$\begin{aligned} 0 &= \mathcal{D}_M[g_1(1+e_{12345}) + ig_2(1-e_{12345})] \\ &= \partial_M g_1(1+e_{12345}) + g_1\mathcal{D}_M(1+e_{12345}) + \mathcal{D}_M[ig_2(1-e_{12345})] \\ &= \partial_M g_1(1+e_{12345}) - g_1\partial_M \log f(1+e_{12345}) + \mathcal{D}_M[ig_2(1-e_{12345})] \ . \end{aligned} \tag{6.4}$$

Multiplying the above equation with g_2^{-1}, we find the Killing spinor equation for η_2 can be expressed as

$$g_2^{-1}[\partial_M(g_1+ig_2) - g_1\partial_M \log f]1 + g_2^{-1}[\partial_M(g_1-ig_2) - g_1\partial_M \log f]e_{12345} + i\mathcal{D}_M(1-e_{12345}) = 0 \ . \tag{6.5}$$

[7]In the normalization of W_5 we have divided with an additional factor of 2 to take into account that in our conventions $\epsilon_{12345} = \sqrt{2}$.

This Killing spinor equation can be analyzed in a way similar to that we have used for the first Killing spinor. In particular, we first express the equation in terms of ten-dimensional data and then write the equation in terms of the Hermitian basis (3.11). A difference between the Killing spinor equation of η_2 (6.4) and that of η_1 is that the last term has an imaginary unit and there is a relative minus sign in the terms involving the holomorphic volume form ϵ. Because of this, it is straightforward to read the conditions arising from Killing spinor equation of η_2 from those of η_1. So we shall not separately give the conditions on the connection and the fluxes arising from the second Killing spinor equation. Instead, we shall proceed to present the analysis of the conditions.

Comparing the Killing spinor equations of η_1 and η_2, we find that the Killing spinor equations that involve the frame time derivative yield the independent equations

$$\partial_0 f = \partial_0(g_1 + ig_2) = 0 \tag{6.6}$$

$$\Omega_{0,\alpha\bar{\beta}} g^{\alpha\bar{\beta}} - \frac{i}{12} F_{\alpha}{}^{\alpha}{}_{\beta}{}^{\beta} = 0 \tag{6.7}$$

$$G_{\alpha_1\alpha_2\alpha_3} = F_{\alpha_1\alpha_2\alpha_3\alpha_4} = 0 \tag{6.8}$$

$$i\Omega_{0,0\bar{\alpha}} + \frac{1}{3} G_{\bar{\alpha}\beta}{}^{\beta} = 0 \tag{6.9}$$

$$\Omega_{0,\bar{\alpha}\bar{\beta}} - \frac{i}{6} F_{\bar{\alpha}\bar{\beta}\gamma}{}^{\gamma} = 0 \ . \tag{6.10}$$

After a comparison between the Killing spinor equations of η_1 and η_2 that involve derivatives along the spatial directions, we find that

$$\partial_{\bar{\alpha}} \log(g_1 f^{-1}) = \partial_{\bar{\alpha}} \log(g_2 f^{-1}) = 0 \ . \tag{6.11}$$

This together with (6.6) imply that there are constants c_1 and c_2 such that $g_1 = c_1 f$ and $g_2 = c_2 f$. Since Killing spinors are specified up to an overall constant scale, without loss of generality, we introduce an angle φ and set $g_1 = \cos\varphi f, g_2 = \sin\varphi f$. The second Killing spinor can be written[8] as

$$\eta_2 = f[\cos\varphi(1 + e_{12345}) + i\sin\varphi(1 - e_{12345})] \ . \tag{6.12}$$

Therefore η_2 is determined by the same spacetime function as η_1. The angle φ is not specified by the Killing spinor equations. However it is required that $\varphi \neq 0, \pi$ because otherwise η_2 will not be linearly independent of η_1.

The rest of the Killing spinor equations for both spinors η_1 and η_2 can then be written as

$$\partial_{\bar{\alpha}} \log f + \frac{1}{2} \Omega_{\bar{\alpha},\beta\bar{\gamma}} g^{\beta\bar{\gamma}} + \frac{i}{12} G_{\bar{\alpha}\gamma}{}^{\gamma} = 0 \ . \tag{6.13}$$

$$\partial_{\bar{\alpha}} \log f - \frac{1}{2} \Omega_{\bar{\alpha},\beta\bar{\gamma}} g^{\beta\bar{\gamma}} + \frac{i}{4} G_{\bar{\alpha}\gamma}{}^{\gamma} = 0 \ . \tag{6.14}$$

$$i\Omega_{\bar{\alpha},0\bar{\beta}} + \frac{1}{6} F_{\bar{\alpha}\bar{\beta}\gamma}{}^{\gamma} = 0 \ . \tag{6.15}$$

[8]This expression is reminiscent of the Killing spinors for dyonic membranes [27].

$$i\Omega_{\bar{\alpha},0\beta} + \frac{1}{12} g_{\bar{\alpha}\beta} F_{\gamma}{}^{\gamma}{}_{\delta}{}^{\delta} + \frac{1}{2} F_{\bar{\alpha}\beta\gamma}{}^{\gamma} = 0 \ . \tag{6.16}$$

$$\Omega_{\bar{\alpha},\bar{\beta}\bar{\gamma}} = 0 \ . \tag{6.17}$$

$$\epsilon_{\bar{\alpha}\bar{\beta}\bar{\gamma}}{}^{\gamma_1\gamma_2} F_{\gamma_1\gamma_2\delta}{}^{\delta} + F_{\bar{\alpha}\gamma_1\gamma_2\gamma_3} \epsilon^{\gamma_1\gamma_2\gamma_3}{}_{\bar{\beta}\bar{\gamma}} = 0 \ . \tag{6.18}$$

$$\Omega_{\bar{\alpha},\beta\gamma} - \frac{i}{2} G_{\bar{\alpha}\beta\gamma} - \frac{i}{3} g_{\bar{\alpha}[\beta} G_{\gamma]\delta}{}^{\delta} = 0 \ . \tag{6.19}$$

These equations can be easily solved to reveal the geometry of spacetime.

6.2 The solution to the Killing spinor equations

The equation (6.8) implies that the $(4,0)+(0,4)$ and $(3,0)+(0,3)$ parts of F and G, respectively, vanish. The equations (6.9) and (6.10) can be easily solved to reveal that

$$G_{\bar{\alpha}\beta}{}^{\beta} = -3i\Omega_{0,0\bar{\alpha}} \tag{6.20}$$

and

$$F_{\bar{\alpha}\bar{\beta}\gamma}{}^{\gamma} = -6i\Omega_{0,\bar{\alpha}\bar{\beta}} \ . \tag{6.21}$$

Next subtract (6.14) from (6.13), we find that

$$G_{\bar{\alpha}\beta}{}^{\beta} = -6i\Omega_{\bar{\alpha},\beta}{}^{\beta} \ . \tag{6.22}$$

Comparing this with (6.20), we have the condition

$$2\Omega_{\bar{\alpha},\beta}{}^{\beta} - \Omega_{0,0\bar{\alpha}} = 0 \ . \tag{6.23}$$

Substituting (6.22) into (6.13), we find $\Omega_{\bar{\alpha},\beta}{}^{\beta} + \partial_{\bar{\alpha}} \log f = 0$. Thus

$$\Omega_{\bar{\alpha},\beta}{}^{\beta} = \frac{1}{2}\Omega_{0,0\bar{\alpha}} = -\partial_{\bar{\alpha}} \log f \ . \tag{6.24}$$

The solution of the equation (6.15) is

$$F_{\bar{\alpha}\bar{\beta}\gamma}{}^{\gamma} = -6i\Omega_{\bar{\alpha},0\bar{\beta}} \tag{6.25}$$

and a comparison with (6.21) reveals that $\Omega_{\bar{\alpha},0\bar{\beta}} = \Omega_{0,\bar{\alpha}\bar{\beta}}$. We shall show that as in the $N=1$ case, the latter condition can be satisfied for an appropriate choice of frame.

Consistency of equation (6.16) with its complex conjugate requires that $\Omega_{\bar{\alpha},0\beta} = -\Omega_{\beta,0\bar{\alpha}}$. This is again satisfied for an appropriate choice of frame. Next take the trace of (6.16) to find

$$F_{\alpha}{}^{\alpha}{}_{\beta}{}^{\beta} = 12i\Omega_{\bar{\alpha},0\beta} g^{\bar{\alpha}\beta} \ , \tag{6.26}$$

which is compatible with (6.7). Substituting back into (6.16), we find that

$$F_{\beta\bar{\alpha}\gamma}{}^{\gamma} = 2i\Omega_{\bar{\alpha},0\beta} - 2i\Omega_{0,\gamma}{}^{\gamma} g_{\bar{\alpha}\beta} \ . \tag{6.27}$$

The equation (6.18) can be solved using (6.25) to give

$$F_{\bar{\alpha}\beta_1\beta_2\beta_3} = 6i g_{\bar{\alpha}[\beta_1} \Omega_{0,\beta_2\beta_3]} \ . \tag{6.28}$$

It remains to investigate equation (6.19). Taking the trace and using (6.22), we find that

$$-\Omega_{\bar\alpha,\gamma}{}^{\bar\alpha} + \Omega_{\gamma,\beta}{}^{\beta} = 0 \ . \tag{6.29}$$

Substituting back into (6.19), we find that

$$G_{\bar\alpha\beta\gamma} = -2i\Omega_{\bar\alpha,\beta\gamma} + 2ig_{\bar\alpha[\beta}\Omega_{0,0\gamma]} \ . \tag{6.30}$$

To summarize, the $(3,0)+(0,3)$ parts of the electric flux G vanish and the (2,1) part is determined in terms of the geometry in (6.30). The $(4,0)+(0,4)$ parts of the magnetic flux F vanish and the (1,3) part is determined in terms of the geometry in (6.28). The trace of the (2,2) part of the magnetic flux is determined in terms of the geometry in (6.27). The traceless part of the (2,2) component of F is not specified by the Killing spinor equations. This concludes the analysis of the solution to the Killing spinor equations.

6.3 The geometry of spacetime

Both Killing spinors give rise to the same Killing vector $\kappa^f = f^2 e^0$. The proof that κ^f is Killing is similar to that in the $N=1$ case and we shall not repeat the calculation. This allows us to adapt coordinates and write the spacetime metric as

$$ds^2 = -f^4(dt+\alpha)^2 + ds^2_{10} \ , \tag{6.31}$$

i.e. as in the $N=1$ case. Consequently, we can find a frame such that $\Omega_{i,0j} = \Omega_{0,ij}$. Thus some of the conditions on the geometry mentioned in the previous section are satisfied.

The remaining conditions are (6.24), (6.29) and (6.17). The latter condition implies that the almost Hermitian ten-dimensional manifold B transverse to the orbits of the Killing vector κ^f is complex, i.e. B is Hermitian. This can be easily seen using the torsion free condition of the Levi-Civita connection. In particular, we have

$$de^{\bar\alpha} = -\Omega_{\beta}{}^{\bar\alpha}{}_{\bar\gamma} e^\beta \wedge e^{\bar\gamma} - \Omega_{\bar\gamma}{}^{\bar\alpha}{}_{\beta} e^{\bar\gamma} \wedge e^\beta - \Omega_{\bar\beta}{}^{\bar\alpha}{}_{\bar\gamma} e^{\bar\beta} \wedge e^{\bar\gamma} \ . \tag{6.32}$$

and therefore the (2,0) part of $de^{\bar\alpha}$ vanishes which implies the integrability of the complex structure. Equivalently, the condition (6.17) implies the vanishing of the Gray-Hervella classes $\mathcal{W}_1, \mathcal{W}_2$, i.e.

$$\mathcal{W}_1 = \mathcal{W}_2 = 0 \ , \tag{6.33}$$

and $\mathrm{w}_1 = \mathrm{w}_2 = 0$. It remains to understand (6.24) and (6.29) in terms of the Gray-Hervella classification. Using (6.29), (6.24) can be expressed as

$$W_5 + 2df = 0 \ , \tag{6.34}$$

which is the condition on the geometry that arises for $N=1$ backgrounds. Therefore it remains to explain (6.29). For this note that

$$(W_4)_\alpha = 2(\mathrm{w}_3)_\alpha = 2\Omega_{\bar\beta,}{}^{\bar\beta}{}_{\alpha} \ , \tag{6.35}$$

where $(W_4)_i = \frac{3}{2}\omega^{jk}\nabla_{[i}\omega_{jk]}$. Using this, we find that (6.29) can be expressed as

$$W_4 - W_5 = 0 \ . \tag{6.36}$$

This concludes the discussion of the geometric conditions arising from the Killing spinor equations with $SU(5)$ invariant spinors.

7 $N = 2$ backgrounds with $SU(4)$ invariant Killing spinors

7.1 The Killing spinor equations

The most general $SU(4)$ invariant Killing spinors of $N = 2$ backgrounds are

$$\begin{aligned}\eta_1 &= f(1 + e_{12345}) \\ \eta_2 &= g_1(1 + e_{12345}) + g_2 i(1 - e_{12345}) + \sqrt{2} g_3(e_5 + e_{1234}) \ .\end{aligned} \tag{7.1}$$

where f, g_1, g_2 and g_3 are real functions of the spacetime which are restricted by the Killing spinor equations. We shall not investigate the most general case here, this will be presented elsewhere [35]. Instead, to simplify the computation we assume that $g_1 = g_2 = 0$ and set $g = g_3 \neq 0$. The Killing spinor equation for η_1 is as in the $N = 1$ case. Multiplying the Killing spinor equation with g^{-1}, we find that

$$\partial_M \log g(e_5 + e_{1234}) + \mathcal{D}_M(e_5 + e_{1234}) = 0 \ . \tag{7.2}$$

To solve the above equation, we first write it in terms of spinors in ten-dimensions and then use the Hermitian basis (3.11) as in the previous cases. Then, because of the form of η_2, the computation of the conditions arising from the Killing spinor equations is most easily done by decomposing the fluxes and the connection in $SU(4)$ representations. In practice this means splitting the holomorphic index $\alpha = (\alpha, 5)$, where now α inside the parenthesis takes values[9] $\alpha = 1, 2, 3, 4$. Using this, the conditions arising from Killing spinor equation for η_2 involving derivatives along the time direction are

$$-i\Omega_{0,05} + \frac{1}{3} G_{5\alpha}{}^{\alpha} - \frac{i}{72} F_{\alpha_1\alpha_2\alpha_3\alpha_4} \epsilon^{\alpha_1\alpha_2\alpha_3\alpha_4} = 0 \ , \tag{7.3}$$

$$\Omega_{0,\bar\alpha 5} + \frac{i}{6} F_{\bar\alpha 5\beta}{}^{\beta} - \frac{1}{18} G_{\beta_1\beta_2\beta_3} \epsilon^{\beta_1\beta_2\beta_3}{}_{\bar\alpha} = 0 \ , \tag{7.4}$$

$$\partial_0 \log g + \frac{1}{2}\Omega_{0,\beta}{}^{\beta} - \frac{1}{2}\Omega_{0,5\bar5} + \frac{i}{24} F_{\alpha}{}^{\alpha}{}_{\beta}{}^{\beta} - \frac{i}{12} F_{\alpha}{}^{\alpha}{}_{5\bar5} = 0 \ , \tag{7.5}$$

$$\frac{1}{6} G_{\bar\alpha\bar\beta 5} + [-\frac{1}{8}\Omega_{0,\gamma_1\gamma_2} - \frac{i}{48} F_{\gamma_1\gamma_2\delta}{}^{\delta} + \frac{i}{48} F_{\gamma_1\gamma_2 5\bar5}] \epsilon^{\gamma_1\gamma_2}{}_{\bar\alpha\bar\beta} = 0 \ , \tag{7.6}$$

$$\frac{i}{36} F_{\beta_1\beta_2\beta_3\bar5} \epsilon^{\beta_1\beta_2\beta_3}{}_{\bar\alpha} - \frac{i}{2}\Omega_{0,0\bar\alpha} + \frac{1}{6} G_{\bar\alpha\gamma}{}^{\gamma} - \frac{1}{6} G_{\bar\alpha 5\bar5} = 0 \ . \tag{7.7}$$

Similarly, the conditions arising from Killing spinor equation for η_2 involving derivatives along the spatial $\bar\alpha$ directions are

$$-i\Omega_{\bar\alpha,05} + \frac{1}{6} F_{\bar\alpha 5\gamma}{}^{\gamma} - \frac{i}{18} g_{\bar\alpha\gamma_1} G_{\gamma_2\gamma_3\gamma_4} \epsilon^{\gamma_1\gamma_2\gamma_3\gamma_4} = 0 \ , \tag{7.8}$$

[9] Throughout this section, $\alpha, \beta, \gamma, \delta, \ldots = 1, 2, 3, 4$.

$$\Omega_{\bar\alpha,\bar\beta 5} - \frac{i}{6}G_{\bar\alpha\bar\beta 5} - (\frac{1}{12}F_{\bar\alpha\gamma_1\gamma_2\gamma_3} + \frac{1}{12}g_{\bar\alpha\gamma_1}F_{\gamma_2\gamma_3\delta}{}^{\delta} - \frac{1}{12}g_{\bar\alpha\gamma_1}F_{\gamma_2\gamma_3 5\bar 5})\epsilon^{\gamma_1\gamma_2\gamma_3}{}_{\bar\beta} = 0 \ , \tag{7.9}$$

$$\partial_{\bar\alpha}\log g + \frac{1}{2}\Omega_{\bar\alpha,\gamma}{}^{\gamma} - \frac{1}{2}\Omega_{\bar\alpha,5\bar 5} - \frac{i}{12}G_{\bar\alpha\gamma}{}^{\gamma} + \frac{i}{12}G_{\bar\alpha 5\bar 5} - \frac{1}{18}g_{\bar\alpha\gamma_1}F_{\gamma_2\gamma_3\gamma_4\bar 5}\epsilon^{\gamma_1\gamma_2\gamma_3\gamma_4} = 0 \ , \tag{7.10}$$

$$\frac{1}{12}F_{\bar\alpha\bar\beta_1\bar\beta_2 5} - \frac{1}{2}(\frac{1}{4}\Omega_{\bar\alpha,\gamma_1\gamma_2} + \frac{i}{8}G_{\bar\alpha\gamma_1\gamma_2} + \frac{i}{12}g_{\bar\alpha\gamma_1}G_{\gamma_2\delta}{}^{\delta} - \frac{i}{12}g_{\bar\alpha\gamma_1}G_{\gamma_2 5\bar 5})\epsilon^{\gamma_1\gamma_2}{}_{\bar\beta_1\bar\beta_2} = 0 \ , \tag{7.11}$$

$$-\frac{i}{2}\Omega_{\bar\alpha,0\bar\beta} + \frac{1}{12}F_{\bar\alpha\bar\beta\gamma}{}^{\gamma} - \frac{1}{12}F_{\bar\alpha\bar\beta 5\bar 5} + \frac{i}{12}\epsilon_{\bar\alpha\bar\beta}{}^{\gamma_1\gamma_2}G_{\gamma_1\gamma_2\bar 5} = 0 \ , \tag{7.12}$$

$$(\frac{i}{2}\Omega_{\bar\alpha,0\gamma} - \frac{1}{4}F_{\bar\alpha\gamma\delta}{}^{\delta} + \frac{1}{4}F_{\bar\alpha\gamma 5\bar 5} - \frac{1}{24}g_{\bar\alpha\gamma}F_{\delta}{}^{\delta}{}_{\sigma}{}^{\sigma} + \frac{1}{12}g_{\bar\alpha\gamma}F_{5\bar 5\delta}{}^{\delta})\epsilon^{\gamma}{}_{\bar\beta_1\bar\beta_2\bar\beta_3} = 0 \ , \tag{7.13}$$

$$\frac{1}{4}\Omega_{\bar\alpha,\bar\beta_1\bar\beta_2} - \frac{i}{24}G_{\bar\alpha\bar\beta_1\bar\beta_2} - \frac{1}{2}(\frac{1}{8}F_{\bar\alpha\bar 5\gamma_1\gamma_2} + \frac{1}{12}g_{\bar\alpha\gamma_1}F_{\gamma_2\bar 5\delta}{}^{\delta})\epsilon^{\gamma_1\gamma_2}{}_{\bar\beta_1\bar\beta_2} = 0 \ , \tag{7.14}$$

$$\partial_{\bar\alpha}\log g - \frac{1}{2}\Omega_{\bar\alpha,\gamma}{}^{\gamma} + \frac{1}{2}\Omega_{\bar\alpha,5\bar 5} - \frac{i}{4}G_{\bar\alpha\gamma}{}^{\gamma} + \frac{i}{4}G_{\bar\alpha 5\bar 5} = 0 \ , \tag{7.15}$$

$$\frac{1}{72}F_{\bar\alpha\bar\beta_1\bar\beta_2\bar\beta_3} + \frac{1}{12}(\frac{1}{2}\Omega_{\bar\alpha,\gamma\bar 5} + \frac{i}{4}G_{\bar\alpha\gamma\bar 5} + \frac{i}{12}g_{\bar\alpha\gamma}G_{\bar 5\delta}{}^{\delta})\epsilon^{\gamma}{}_{\bar\beta_1\bar\beta_2\bar\beta_3} = 0 \ , \tag{7.16}$$

$$\frac{i}{2}\Omega_{\bar\alpha,0\bar 5} - \frac{1}{4}F_{\bar\alpha\bar 5\gamma}{}^{\gamma} = 0 \ . \tag{7.17}$$

The conditions arising from Killing spinor equation for η_2 involving derivatives along the spatial $\bar 5$ direction are

$$i\Omega_{\bar 5,05} + \frac{1}{12}F_{\gamma}{}^{\gamma}{}_{\delta}{}^{\delta} + \frac{1}{3}F_{5\bar 5\gamma}{}^{\gamma} = 0 \ , \tag{7.18}$$

$$\Omega_{\bar 5,\bar\beta 5} - \frac{i}{6}G_{\bar\beta\gamma}{}^{\gamma} - \frac{i}{3}G_{\bar\beta 5\bar 5} - \frac{1}{36}F_{\bar 5\gamma_1\gamma_2\gamma_3}\epsilon^{\gamma_1\gamma_2\gamma_3}{}_{\bar\beta} = 0 \ , \tag{7.19}$$

$$\partial_{\bar 5}\log g + \frac{1}{2}\Omega_{\bar 5,\gamma}{}^{\gamma} - \frac{1}{2}\Omega_{\bar 5,5\bar 5} - \frac{i}{4}G_{\bar 5\gamma}{}^{\gamma} = 0 \ , \tag{7.20}$$

$$\frac{1}{12}F_{\bar\beta_1\bar\beta_2\gamma}{}^{\gamma} + \frac{1}{6}F_{\bar\beta_1\bar\beta_2 5\bar 5} + \frac{1}{2}(\frac{1}{4}\Omega_{\bar 5,\gamma_1\gamma_2} + \frac{i}{24}G_{\bar 5\gamma_1\gamma_2})\epsilon^{\gamma_1\gamma_2}{}_{\bar\beta_1\bar\beta_2} = 0 \ , \tag{7.21}$$

$$-\frac{i}{2}\Omega_{\bar{5},0\bar{\beta}}-\frac{1}{4}F_{\bar{\beta}\bar{5}\gamma}{}^{\gamma}=0\ , \tag{7.22}$$

$$-\frac{i}{36}G_{\bar{\beta}_1\bar{\beta}_2\bar{\beta}_3}+\frac{1}{12}(\frac{i}{2}\Omega_{\bar{5},0\gamma}-\frac{1}{12}F_{\bar{5}\gamma\delta}{}^{\delta})\epsilon^{\gamma}{}_{\bar{\beta}_1\bar{\beta}_2\bar{\beta}_3}=0\ , \tag{7.23}$$

$$\Omega_{\bar{5},\bar{\beta}_1\bar{\beta}_2}-\frac{i}{2}G_{\bar{5}\bar{\beta}_1\bar{\beta}_2}=0\ , \tag{7.24}$$

$$-\frac{1}{144}F_{\bar{\beta}_1\bar{\beta}_2\bar{\beta}_3\bar{\beta}_4}+\frac{1}{96}(\partial_{\bar{5}}\log g-\frac{1}{2}\Omega_{\bar{5},\gamma}{}^{\gamma}+\frac{1}{2}\Omega_{\bar{5},5\bar{5}}-\frac{i}{12}G_{\bar{5}\gamma}{}^{\gamma})\epsilon_{\bar{\beta}_1\bar{\beta}_2\bar{\beta}_3\bar{\beta}_4}=0\ , \tag{7.25}$$

$$-F_{\bar{\beta}_1\bar{\beta}_2\bar{\beta}_3\bar{5}}+\Omega_{\bar{5},\gamma\bar{5}}\epsilon^{\gamma}{}_{\bar{\beta}_1\bar{\beta}_2\bar{\beta}_3}=0\ . \tag{7.26}$$

7.2 Solution to the Killing spinor equations

To solve the Killing spinor equations for η_2, we use the solutions to the Killing spinor equations for η_1. The latter relate certain components of the fluxes to spacetime geometry as represented by the components of the spin connection. We substitute all these relations into the conditions that arise from the Killing spinor equations for η_2. As a result, we turn the conditions associated with the Killing spinor equations for η_2 into conditions on the geometry of spacetime. The calculation is long but routine and it is explained in detail in appendix D. Here we summarize the conditions that arise from the analysis.

The Killing spinor equations imply that

$$g=f\ ,\quad \partial_0 f=(\partial_5-\partial_{\bar{5}})f=0\ . \tag{7.27}$$

In fact g is proportional to f but as we have mentioned Killing spinors are determined up to a constant scale. The conditions on the $\Omega_{0,0i}$ components are

$$\Omega_{0,05}=\Omega_{0,0\bar{5}}=-2\partial_5\log f=-2\partial_{\bar{5}}\log f\ ,\qquad \Omega_{0,0\alpha}=-2\partial_\alpha\log f\ . \tag{7.28}$$

The conditions on the $\Omega_{0,ij}$ components are

$$\Omega_{0,5\bar{\alpha}}=\Omega_{0,5\alpha}=\Omega_{0,5\bar{5}}=\Omega_{0,\beta}{}^{\beta}=0\ ,\qquad \Omega_{0,\beta_1\beta_2}=\frac{i}{4}(\Omega_{5,\bar{\gamma}_1\bar{\gamma}_2}-\Omega_{\bar{5},\bar{\gamma}_1\bar{\gamma}_2})\epsilon^{\bar{\gamma}_1\bar{\gamma}_2}{}_{\beta_1\beta_2} \tag{7.29}$$

and the traceless part of $\Omega_{0,\alpha\bar{\beta}}$ is not determined. The conditions on the $\Omega_{\bar{\alpha},ij}$ components are

$$\begin{aligned}\Omega_{[\bar{\beta}_1,\bar{\beta}_2\bar{\beta}_3]} &= 0\ ,\quad \Omega_{\bar{\alpha},\beta_1\beta_2}=-\Omega_{0,0[\beta_1}g_{\beta_2]\bar{\alpha}}\ ,\quad \Omega_{\beta,\bar{\alpha}}{}^{\beta}=\frac{3}{2}(\Omega_{\bar{5},\bar{\alpha}\bar{5}}-\Omega_{\bar{5},\bar{\alpha}5})=-\frac{3}{2}\Omega_{0,0\bar{\alpha}}\ ,\\ \Omega_{\alpha,\beta}{}^{\beta} &= -\frac{1}{2}(\Omega_{\bar{5},\alpha\bar{5}}+\Omega_{\bar{5},\alpha5})=-\frac{1}{2}(\Omega_{0,0\alpha}+2\Omega_{5,\alpha5})\ .\end{aligned} \tag{7.30}$$

In addition, we have

$$\Omega_{[\bar{\beta}_1,\bar{\beta}_2]\bar{5}}=-\Omega_{\bar{5},\bar{\beta}_1\bar{\beta}_2}\ ,\quad \Omega_{[\bar{\beta}_1,\bar{\beta}_2]5}=-\Omega_{5,\bar{\beta}_1\bar{\beta}_2}\ ,\quad \Omega_{(\bar{\beta}_1,\bar{\beta}_2)5}=\Omega_{(\bar{\beta}_1,\bar{\beta}_2)\bar{5}}\ ,$$

$$
\Omega_{(\bar{\alpha},\beta)\bar{5}} = \Omega_{(\bar{\alpha},\beta)5} = \frac{1}{2} g_{\bar{\alpha}\beta}\Omega_{0,0\bar{5}} \ , \quad \Omega_{\bar{\alpha},5\bar{5}} = 0 \ . \tag{7.31}
$$

Finally, the conditions on the $\Omega_{\bar{5},ij}$ components are

$$
\begin{aligned}
&\Omega_{5,\beta}{}^{\beta} = \Omega_{\bar{5},\beta}{}^{\beta} \ , \quad \Omega_{5,\bar{\alpha}5} = \Omega_{\bar{5},\bar{\alpha}\bar{5}} \ , \quad \Omega_{5,\bar{\alpha}\bar{5}} = \Omega_{\bar{5},\bar{\alpha}5} \ , \\
&\Omega_{\bar{5},\bar{\alpha}\bar{5}} - \Omega_{\bar{5},\bar{\alpha}5} = -\Omega_{0,0\bar{\alpha}} \ , \quad \Omega_{\bar{5},5\bar{5}} = -\Omega_{5,5\bar{5}} = -\Omega_{0,0\bar{5}} \ .
\end{aligned} \tag{7.32}
$$

The above equations together with their complex conjugates give full set of conditions that are required for a background to admit $N = 2$ supersymmetry and Killing spinors given by $\eta_1 = f\eta^{SU(5)}$ and $\eta_2 = g\eta^{SU(5)}$. The traceless part of $\Omega_{\alpha,\beta\bar{\gamma}}$ is not determined by the Killing spinor equations.

7.3 Fluxes

The conditions we have derived for the spin connection in turn restrict the form of the fluxes. In particular we find that the electric part of the flux is

$$
\begin{aligned}
&G_{\alpha\beta\gamma} = 0 \ , \quad G_{5\beta\gamma} = 2i\Omega_{5,\beta\gamma} \ , \quad G_{\bar{\alpha}5\gamma} = -2i\Omega_{\bar{\alpha},5\gamma} - ig_{\bar{\alpha}\gamma}\Omega_{0,05} \ , \\
&G_{\bar{5}\beta\gamma} = -2i\Omega_{\bar{5},\beta\gamma} \ , \quad G_{\bar{\alpha}\beta\gamma} = 0 \ , \quad G_{\bar{5}5\alpha} = -2i\Omega_{\bar{5},5\alpha} + i\Omega_{0,0\alpha}
\end{aligned} \tag{7.33}
$$

Similarly, the magnetic part of the flux is

$$
\begin{aligned}
&F_{\alpha_1\alpha_2\alpha_3\alpha_4} = \frac{1}{2}(-3\Omega_{0,05} + 2\Omega_{5,\beta}{}^{\beta})\epsilon_{\alpha_1\alpha_2\alpha_3\alpha_4} \ , \quad F_{5\alpha_1\alpha_2\alpha_2} = \frac{1}{2}(\Omega_{0,0\bar{\beta}} - 2\Omega_{\bar{\beta},\gamma}{}^{\gamma})\epsilon^{\bar{\beta}}{}_{\alpha_1\alpha_2\alpha_3} \\
&F_{\bar{\alpha}\beta_1\beta_2\beta_3} = \frac{1}{2}[2\Omega_{\bar{\alpha},\bar{5}\bar{\gamma}}\epsilon^{\bar{\gamma}}{}_{\beta_1\beta_2\beta_3} - 3\Omega_{5,\bar{\gamma}_1\bar{\gamma}_2}\epsilon^{\bar{\gamma}_1\bar{\gamma}_2}{}_{[\beta_1\beta_2}g_{\beta_3]\bar{\alpha}}] \ , \quad F_{\bar{5}\beta_1\beta_2\beta_3} = -\Omega_{\bar{5},\bar{\gamma}\bar{5}}\epsilon^{\bar{\gamma}}{}_{\beta_1\beta_2\beta_3} \\
&F_{\bar{\alpha}5\beta_1\beta_2} = \frac{1}{2}\Omega_{\bar{\alpha},\bar{\gamma}_1\bar{\gamma}_2}\epsilon^{\bar{\gamma}_1\bar{\gamma}_2}{}_{\beta_1\beta_2} \ , \quad F_{5\bar{5}\alpha}{}^{\alpha} = 0 \ , \quad F_{\alpha\bar{\beta}\gamma}{}^{\gamma} = 0 \\
&F_{\alpha\beta5\bar{5}} = \frac{1}{2}(\Omega_{5,\bar{\gamma}_1\bar{\gamma}_2} - \Omega_{\bar{5},\bar{\gamma}_1\bar{\gamma}_2})\epsilon^{\bar{\gamma}_1\bar{\gamma}_2}{}_{\alpha\beta} \ , \quad F_{\alpha\bar{\beta}5\bar{5}} = -2i\Omega_{0,\alpha\bar{\beta}} \ , \\
&F_{\bar{\alpha}\bar{5}\beta_1\beta_2} = \frac{1}{2}\Omega_{\bar{\alpha},\bar{\gamma}_1\bar{\gamma}_2}\epsilon^{\bar{\gamma}_1\bar{\gamma}_2}{}_{\beta_1\beta_2}
\end{aligned} \tag{7.34}
$$

The last two relations are derived from the conditions for $N = 2$ supersymmetry (D.12) and (D.13) in appendix D, respectively. The components of the fluxes that do not appear in the above equations are not determined by the Killing spinor equations.

7.4 The geometry of spacetime

We shall now investigate some aspects of the spacetime geometry that arises from the relations (7.27)-(7.32). The spacetime admits a timelike Killing vector field κ^f which is inherited from the Killing spinor η_1 as in the $N = 1$ case. From the forms associated with the spinors $\eta^{SU(5)}$ and $\eta^{SU(4)}$, it is clear that the spacetime has an $SU(4)$-structure. In particular, the space B transverse to the orbits of κ^f has an $SU(4)$-structure. The space B is not complex because the almost complex structure associated with $\omega(\eta^{SU(5)}, \eta^{SU(5)})$ is *not integrable* as can be seen by looking at the components of the spin connection. In particular the component $\Omega_{\bar{5},\bar{\alpha}\bar{5}}$ of the connection is not required to vanish by the Killing

spinor equations. This is a difference between the $N=2$ backgrounds with $SU(5)$ and $SU(4)$ invariant structures. In the former case, B is complex.

As we have seen, there is a one-form (4.19) constructed from the spinors $\eta^{SU(5)}$ and $\eta^{SU(4)}$. Using this, we can define the vector field

$$\tilde{\kappa}^f = f^2\partial_\natural = if^2(\partial_5 - \partial_{\bar{5}}) \ . \tag{7.35}$$

It turns out that this is a Killing vector field on the spacetime. To show this, we have to show that $\tilde{\kappa}^f$ solves the Killing vector equation, $\nabla_A\tilde{\kappa}^f_B + \nabla_B\tilde{\kappa}^f_A = 0$. In terms of the connection, this can be written as

$$\partial_A f^2(\tilde{\kappa})_B + \partial_B f^2(\tilde{\kappa})_A - \Omega_{A,}{}^C{}_B(\tilde{\kappa}^f)_C - \Omega_{B,}{}^C{}_A(\tilde{\kappa}^f)_C = 0 \ , \tag{7.36}$$

where the non-vanishing components of the associated one-form to the vector field are $(\tilde{\kappa})_5 = -(\tilde{\kappa})_{\bar{5}} = -i$ and $\tilde{\kappa}^f = f^2\tilde{\kappa}$. This equation can be easily verified using the equations (7.27)-(7.32). For example setting $(A,B) = (0,0)$ we get, by using that f is time-independent (7.27), that

$$2i\Omega_{0,}{}^5{}_0 - 2i\Omega_{0,}{}^{\bar{5}}{}_0 = 0 \tag{7.37}$$

which vanishes due to (7.28). In a similar way, one can show that $\tilde{\kappa}^f$ is a Killing vector using the rest of the conditions.

In fact $\tilde{\kappa}^f$ preserves the almost complex structure as well, i.e. $\mathcal{L}_{\tilde{\kappa}^f}\omega(\eta^{SU(4)},\eta^{SU(4)}) = 0$. The computation can be simplified by expressing the Lie derivative in terms of the spin connection as

$$(\mathcal{L}_{\tilde{\kappa}^f}\omega)_{AB} = 2\partial_{[A}f^2\omega_{|C|B]}(\tilde{\kappa})^C - 2(\tilde{\kappa}^f)^D\Omega_{[D,}{}^C{}_{A]}\omega_{CB} + 2(\tilde{\kappa}^f)^D\Omega_{[D,}{}^C{}_{B]}\omega_{CA} \tag{7.38}$$

where $\omega = \omega(\eta^{SU(5)},\eta^{SU(5)})$. Then one can proceed to verify the equation $\mathcal{L}_{\tilde{\kappa}^f}\omega = 0$ using the condition (7.27)-(7.32). In fact κ^f also preserves the almost complex structure of B. It is likely that both κ^f and $\tilde{\kappa}^f$ preserve the whole of the $SU(4)$ structure of B including the higher degree forms that are associated with the spinors $\eta^{SU(5)}$ and $\eta^{SU(4)}$.

In addition, one can show that $[\kappa^f,\tilde{\kappa}^f] = 0$. This is because all the components of the connection of the type $\Omega_{0,5i} = 0$ and also due to (7.27). In such a case, one can introduce coordinates u^a adapted to both Killing vector fields and write the metric as

$$ds^2 = U_{ab}(du^a + \beta^a)(du^b + \beta^b) + \gamma_{IJ}dx^I dx^J \ , \tag{7.39}$$

where U_{ab}, β and γ depend on the remaining coordinates x^I.

To summarize, we have shown that the $N=2$ backgrounds with $SU(4)$ invariant spinors admit two commuting Killing vector fields κ^f and $\tilde{\kappa}^f$. The former is timelike while the latter is spacelike. The space B transverse to the orbits of κ^f is an almost Hermitian manifold with an $SU(4)$-structure with a Killing vector field which preserves the almost complex structure. The $SU(4)$-structure on B is determined by the conditions (7.28)-(7.32), see also appendix B. Of course these conditions can be put into real form using the almost complex structure and the forms associated with the spinors.

8 $N > 2$ backgrounds

8.1 General case

To investigate backgrounds with more than two supersymmetries, one can repeat the procedure that we have used for the $N = 2$ case. Suppose that we have chosen the first two Killing spinors to be in the directions η_1 and η_2 with η_1 representing the orbit $\mathcal{O}_{SU(5)}$. To choose the third Killing spinor, we decompose Δ^+_{16} under the action of the stability subgroup $H \subset SU(5)$ that leaves both η_1 and η_2 invariant. Typically H is an $SU(n)$ or a product of $SU(n)$ groups. Then, we look at the orbits of H in Δ^+_{16}. Using the results in appendix A, it is straightforward to find these orbits. The third Killing spinor η_3 can be chosen as a linear combination of the representatives of these orbits and linearly independent of η_1 and η_2.

Clearly this procedure can be repeated to find representatives for any number of Killing spinors. This method works well in the cases that the stability subgroup of the spinors is large because it can be used to restrict the choice of the next spinor and to solve the Killing spinor equations. In the case that the stability subgroup is small, further progress depends on the details of the Killing spinor equation. For example, suppose that η_1 and η_2 are chosen such that the stability subgroup is $\{1\}$. If this is the case, the third spinor can be chosen as any other spinor which is linearly independent of η_1 and η_2. Although our formalism can still be used, there is no apparent simplification in the computation of the Killing spinor equations and the forms associated with η_3. As a result, the geometry of the background will be rather involved. Because of this, we shall focus on those $N > 2$ backgrounds which admit spinors with large symmetry groups. Amongst the various cases, there is one for which the spinors are invariant under SU groups. To illustrate the general procedure of constructing canonical forms for the Killing spinors outlined above, we will analyze this case further below.

8.2 The SU series

The SU series is characterized by the property that the Killing spinors are progressively invariant under $1 \subset SU(2) \subset SU(3) \subset SU(4) \subset SU(5)$. This series can also be thought of as the Calabi-Yau series. The Killing spinors that we give below are those expected in M-theory Calabi-Yau compactifications (with fluxes).

We begin by choosing $\eta_1 = \eta^{SU(5)}$ and $\eta_2 = \theta^{SU(5)}$[10] which are invariant under $SU(5)$. There are no other linearly independent spinors invariant under $SU(5)$. To find the spinors invariant under $SU(4) \subset SU(5)$, recall that we have analyzed all possible choices of spinors which are linearly independent from η_1 and η_2. We have found that the only $SU(4)$ invariant spinors are $\eta^{SU(4)}$ and $\theta^{SU(4)}$. Therefore we choose[11]

$$\eta_3 = \eta^{SU(4)} \,, \tag{8.1}$$

[10] We could have taken η_2 to be a linear combination of $\eta^{SU(5)}$ and $\theta^{SU(5)}$ without changing the analysis below but for simplicity we have set $\eta_2 = \theta^{SU(5)}$.

[11] Again, for η_3 and η_4 we could have chosen a linear combination of $SU(4)$ and $SU(5)$ invariant spinors but for simplicity we have not done so. Similar considerations arise below for spinors invariant under other SU groups.

and

$$\eta_4 = \theta^{SU(4)} \ . \tag{8.2}$$

The stability subgroup of all the spinors $\eta_1, \dots, \eta_4$ is $SU(4)$. Therefore there are four $SU(4)$ invariant spinors which is the same number of spinors as those expected for a compactification of M-theory on an eight-dimensional Calabi-Yau manifold. To proceed, we decompose Δ^+_{16} under $SU(4)$. Of course $\Lambda^0(\mathbb{C}^5)$ does not decompose further. The one and three forms decompose as

$$\begin{aligned} \Lambda^1_5(\mathbb{C}^5) &= \Lambda^0(\mathbb{C}^5) \oplus \Lambda^1_4(\mathbb{C}^4) \ , \\ \Lambda^3_{10} &= \Lambda^2_6(\mathbb{C}^4) \oplus \Lambda^3_4(\mathbb{C}^4) \ . \end{aligned} \tag{8.3}$$

Using the results of appendix A, it is easy to see that the only additional $SU(3)$ invariant spinors are

$$e_4 \ , \qquad e_{123} \ . \tag{8.4}$$

Each of these gives two Majorana spinors. Therefore, we find four additional spinors which are invariant under $SU(3)$. These are

$$\begin{aligned} \eta^{SU(3)}_1 &= \frac{1}{\sqrt{2}}(e_4 - e_{1235}) \ , \qquad \theta^{SU(3)}_1 = \frac{i}{\sqrt{2}}(e_4 + e_{1235}) \ , \\ \eta^{SU(3)}_2 &= \frac{1}{\sqrt{2}}(e_{45} - e_{123}) \ , \qquad \theta^{SU(3)}_2 = \frac{i}{\sqrt{2}}(e_{45} + e_{123}) \ . \end{aligned} \tag{8.5}$$

Therefore there are eight $SU(3)$ invariant spinors, i.e. as many as the number of Killing spinors expected for a compactification of M-theory on a six-dimensional Calabi-Yau manifold.

To proceed, we decompose $\Lambda^1_4(\mathbb{C}^4)$, $\Lambda^2_6(\mathbb{C}^4)$ and $\Lambda^3_4(\mathbb{C}^4)$ under $SU(3)$ to find

$$\begin{aligned} \Lambda^1_4(\mathbb{C}^4) &= \Lambda^0(\mathbb{C}^3) \oplus \Lambda^1_3(\mathbb{C}^3) \ , \\ \Lambda^3_4(\mathbb{C}^4) &= \Lambda^2(\mathbb{C}^3) \oplus \Lambda^3_1(\mathbb{C}^3) \ , \\ \Lambda^2_6(\mathbb{C}^4) &= \Lambda^1_3(\mathbb{C}^3) \oplus \Lambda^2_3(\mathbb{C}^3) \ . \end{aligned} \tag{8.6}$$

It is easy to see that the additional $SU(2)$ invariant complex spinors are

$$e_3 \ , \quad e_{345} \ , \quad e_{124} \ , \quad e_{125} \tag{8.7}$$

Each of these complex spinors gives rise to two Majorana spinors which are given by

$$\begin{aligned} \eta^{SU(2)}_1 &= \frac{1}{\sqrt{2}}(e_3 + e_{1245}) \ , \qquad \theta^{SU(2)}_1 = \frac{i}{\sqrt{2}}(e_3 - e_{1245}) \ , \\ \eta^{SU(2)}_2 &= \frac{1}{\sqrt{2}}(e_{12} - e_{345}) \ , \qquad \theta^{SU(2)}_2 = \frac{i}{\sqrt{2}}(e_{12} + e_{345}) \ , \\ \eta^{SU(2)}_3 &= \frac{1}{\sqrt{2}}(e_{35} + e_{124}) \ , \qquad \theta^{SU(2)}_3 = \frac{i}{\sqrt{2}}(e_{35} - e_{124}) \ , \\ \eta^{SU(2)}_4 &= \frac{1}{\sqrt{2}}(e_{34} - e_{125}) \ , \qquad \theta^{SU(2)}_4 = \frac{i}{\sqrt{2}}(e_{34} + e_{125}) \ . \end{aligned} \tag{8.8}$$

The total number of $SU(2)$ invariant spinors is sixteen, i.e. as many as those expected for an M-theory compactification on K_3. Giving here all the spinors explicitly allows one to construct all the forms associated with these spinors and in this way get an insight into the geometry of the supersymmetric background. We shall not present the results of the Killing spinor analysis here. These can be found elsewhere [35].

9 $N = 3$ backgrounds with $SU(4)$ invariant spinors

To investigate a class of $N = 3$ backgrounds it suffices to combine the conditions we have derived for $N = 2$ backgrounds with $SU(5)$ and $SU(4)$ invariant spinors. Such $N = 3$ backgrounds have Killing spinors $\eta_1 = f_1\eta^{SU(5)}$, $\eta_2 = f_2\theta^{SU(5)}$ and $\eta_3 = f_3\eta^{SU(4)}$. Combining the conditions of the two classes of $N = 2$ backgrounds, we find that

$$f_1 = f_2 = f_3 = f \ , \quad \partial_0 f = \partial_5 f = \partial_{\bar 5} f = 0 \ . \tag{9.1}$$

Further combining the condition that the $(3,0) + (0,3)$ parts of $\Omega_{i,jk}$ vanish with the conditions summarized in section 7.2, we find that the conditions on the $\Omega_{0,0i}$ components are

$$\Omega_{0,05} = \Omega_{0,0\bar 5} = 0 \ , \qquad \Omega_{0,0\alpha} = -2\partial_\alpha \log f \ . \tag{9.2}$$

The conditions on the $\Omega_{0,ij}$ components are

$$\Omega_{0,5\bar\alpha} = \Omega_{0,5\alpha} = \Omega_{0,5\bar 5} = \Omega_{0,\beta}{}^{\beta} = 0 \ , \qquad \Omega_{0,\beta_1\beta_2} = \frac{i}{4}\Omega_{5,\bar\gamma_1\bar\gamma_2}\epsilon^{\bar\gamma_1\bar\gamma_2}{}_{\beta_1\beta_2} \tag{9.3}$$

and the traceless part of $\Omega_{0,\alpha\bar\beta}$ is not determined. The conditions on the $\Omega_{\bar\alpha,ij}$ components are

$$\begin{aligned} \Omega_{\bar\beta_1,\bar\beta_2\bar\beta_3} &= 0 \ , \quad \Omega_{\bar\alpha,\beta_1\beta_2} = -\Omega_{0,0[\beta_1}g_{\beta_2]\bar\alpha} \ , \quad \Omega_{\beta,\bar\alpha}{}^{\beta} = -\frac{3}{2}\Omega_{\bar 5,\bar\alpha 5} = -\frac{3}{2}\Omega_{0,0\bar\alpha} \ , \\ \Omega_{\alpha,\beta}{}^{\beta} &= -\frac{1}{2}\Omega_{\bar 5,\alpha 5} = -\frac{1}{2}\Omega_{0,0\alpha} \ . \end{aligned} \tag{9.4}$$

In addition, we have

$$\begin{aligned} &\Omega_{\bar\beta_1,\bar\beta_2\bar 5} = \Omega_{\bar 5,\bar\beta_1\bar\beta_2} = 0 \ , \quad \Omega_{[\bar\beta_1,\bar\beta_2]5} = -\Omega_{5,\bar\beta_1\bar\beta_2} \ , \quad \Omega_{(\bar\beta_1,\bar\beta_2)5} = 0 \ , \\ &\Omega_{(\bar\alpha,\beta)\bar 5} = \Omega_{(\bar\alpha,\beta)5} = 0 \ , \quad \Omega_{\bar\alpha,5\bar 5} = 0 \ . \end{aligned} \tag{9.5}$$

The traceless part of $\Omega_{\alpha,\beta\bar\gamma}$ is not determined. Finally, the conditions on the $\Omega_{\bar 5,ij}$ components are

$$\begin{aligned} &\Omega_{5,\beta}{}^{\beta} = \Omega_{\bar 5,\beta}{}^{\beta} = 0 \ , \quad \Omega_{5,\bar\alpha 5} = \Omega_{\bar 5,\bar\alpha\bar 5} = 0 \ , \quad \Omega_{5,\bar\alpha\bar 5} = \Omega_{\bar 5,\bar\alpha 5} \ , \\ &\Omega_{\bar 5,\bar\alpha 5} = \Omega_{0,0\bar\alpha} \ , \quad \Omega_{\bar 5,5\bar 5} = \Omega_{5,5\bar 5} = 0 \ . \end{aligned} \tag{9.6}$$

The above equations together with their complex conjugates give full set of conditions that are required for a background to admit $N = 3$ supersymmetry and Killing spinors given by $\eta_1 = f_1\eta^{SU(5)}$, $\eta_2 = f_2\theta^{SU(5)}$ and $\eta_3 = f_3\eta^{SU(4)}$.

9.1 Fluxes

We can substitute the conditions on the geometry that we have derived into the expressions for the fluxes. As a result, the electric part of the flux can be written as

$$\begin{aligned} &G_{\alpha\beta\gamma} = 0 \ , \quad G_{5\beta\gamma} = 0 \ , \quad G_{\bar\alpha 5\gamma} = -2i\Omega_{\bar\alpha,5\gamma} \ , \\ &G_{\bar 5\beta\gamma} = -2i\Omega_{\bar 5,\beta\gamma} \ , \quad G_{\bar\alpha\beta\gamma} = 0 \ , \quad G_{\bar 5 5\alpha} = -2i\Omega_{\bar 5,5\alpha} + i\Omega_{0,0\alpha} \end{aligned} \tag{9.7}$$

Similarly, the magnetic part of the flux is

$$
\begin{aligned}
&F_{\bar{\alpha}\beta_1\beta_2\beta_3} = -\frac{3}{2}\Omega_{5,\bar{\gamma}_1\bar{\gamma}_2}\epsilon^{\bar{\gamma}_1\bar{\gamma}_2}{}_{[\beta_1\beta_2}g_{\beta_3]\bar{\alpha}} \ , \quad F_{\alpha_1\alpha_2\alpha_3\alpha_4} = F_{5\alpha_1\alpha_2\alpha_2} = F_{\bar{5}\beta_1\beta_2\beta_3} = 0 \ , \\
&F_{\bar{\alpha}5\beta_1\beta_2} = 0 \ , \quad F_{5\bar{5}\alpha}{}^{\alpha} = 0 \ , \quad F_{\alpha\bar{\beta}\gamma}{}^{\gamma} = 0 \ , \quad F_{\alpha\beta 5\bar{5}} = \frac{1}{2}\Omega_{5,\bar{\gamma}_1\bar{\gamma}_2}\epsilon^{\bar{\gamma}_1\bar{\gamma}_2}{}_{\alpha\beta} \ , \\
&F_{\alpha\bar{\beta}5\bar{5}} = -2i\Omega_{0,\alpha\bar{\beta}} \ , \quad F_{\bar{\alpha}\bar{5}\beta_1\beta_2} = 0 \ .
\end{aligned} \tag{9.8}
$$

We have used that the (3,0)+(0,3) part of the connection $\Omega_{i,jk}$ vanishes. The last two relations are derived from the conditions for $N=2$ supersymmetry (D.12) and (D.13) in appendix D, respectively. The components of the fluxes that do not appear in the above equations are not determined by the Killing spinor equations.

9.2 Geometry

We shall now investigate some aspects of the spacetime geometry that arises from the relations (9.1)-(9.6). The geometry of these $N=3$ backgrounds combines aspects of the geometries of the $N=2$ backgrounds with $SU(5)$ and $SU(4)$ invariant Killing spinors that we have investigated. As in all previous cases, the spacetime admits a timelike Killing vector field κ^f which is inherited from the Killing spinor η_1. From the forms associated with the spinors $\eta^{SU(5)}$, $\theta^{SU(5)}$ and $\eta^{SU(4)}$, it is clear that the spacetime admits an $SU(4)$-structure. In particular, the space B transverse to the orbits of κ^f has an $SU(4)$-structure. However, unlike the $N=2$ backgrounds with $SU(4)$ invariant spinors, the space B is complex and therefore Hermitian. This is because the (3,0)+(0,3) parts of the connection $\Omega_{A,BC}$ vanish. This is similar to the $N=2$ backgrounds with $SU(5)$ invariant spinors.

There are also two spacelike Killing vector fields $\tilde{\kappa}^f = if^2(\partial_5-\partial_{\bar{5}})$ and $\hat{\kappa}^f = f^2(\partial_5+\partial_{\bar{5}})$ which are associated with the one-forms (4.19) and (4.25), respectively, constructed from the spinors $\eta^{SU(5)}$, $\theta^{SU(5)}$ and $\eta^{SU(4)}$. The Killing vector fields κ^f and $\tilde{\kappa}^f$ and $\hat{\kappa}^f$ commute. The proof of both these statements is similar to those of $N=2$ backgrounds with $SU(4)$ invariant spinors.

The $\tilde{\kappa}^f$ and $\hat{\kappa}^f$ preserve the complex structure of B as well, e.g. $\mathcal{L}_{\tilde{\kappa}^f}\omega(\eta^{SU(5)},\eta^{SU(5)}) = 0$. The computation is similar to that we presented for the $N=2$ backgrounds with $SU(4)$ invariant Killing spinors. It is also likely that all three vector fields κ^f, $\hat{\kappa}^f$ and $\tilde{\kappa}^f$ preserve the whole of the $SU(4)$ structure of B including the higher degree forms that are associated with the spinors $\eta^{SU(5)}$, $\theta^{SU(5)}$ and $\eta^{SU(4)}$.

The space $\hat{B}$ transverse to the orbits of all three vector fields is Hermitian with respect to the complex structure associated with the two-form $\omega^{SU(4)}$, see (4.24). In addition $\hat{B}$ admits an $SU(4)$-structure. We can adapt coordinates to all the above vector fields and write the metric as

$$
ds^2 = U_{ab}(du^a+\beta^a)(du^b+\beta^b)+\gamma_{IJ}dx^I dx^J \tag{9.9}
$$

where U, β and γ are functions of the x coordinates, and $a,b=0,1,2$ and $I,J=1,\dots,8$.

To summarize, we have shown that the $N=3$ backgrounds with $SU(4)$ invariant spinors admit three commuting Killing vector fields κ^f, $\hat{\kappa}^f$ and $\tilde{\kappa}^f$. The former is timelike while the other two are spacelike. The space B transverse to the orbits of κ^f is a

Hermitian manifold with an $SU(4)$-structure and admits two holomorphic Killing vector fields. The $SU(4)$-structure on B is determined by the conditions (9.2)-(9.6), see also appendix B. Of course these conditions can be put into real form using the complex structure and the forms associated with the spinors.

10 $N=4$ backgrounds with $SU(4)$ invariant spinors

We can also easily investigate a class of $N=4$ backgrounds, namely $N=4$ backgrounds that have the Killing spinors $\eta_1 = f_1\eta^{SU(5)}$, $\eta_2 = f_2\theta^{SU(5)}$, $\eta_3 = f_3\eta^{SU(4)}$ and $\eta_4 = f_4\theta^{SU(4)}$, where f_1, f_2, f_3 and f_4 are real functions of the spacetime. The conditions coming from the three first Killing spinors have already been computed in the previous subsection, and the conditions from η_4 can be obtained from the formulas in section 7.1 by changing signs on all terms containing the epsilon tensor. For this compare the expressions for $\eta^{SU(4)}$ and $\theta^{SU(4)}$. The conditions we find from the Killing spinor equations are

$$f_1 = f_2 = f_3 = f_4 = f \ , \quad \partial_0 f = \partial_5 f = \partial_{\bar{5}} f = 0 \ . \tag{10.1}$$

Furthermore, the conditions on the $\Omega_{0,0i}$ components are

$$\Omega_{0,05} = \Omega_{0,0\bar{5}} = -2\partial_5 \log f = -2\partial_{\bar{5}} \log f = 0 \ , \qquad \Omega_{0,0\alpha} = -2\partial_\alpha \log f \ . \tag{10.2}$$

The conditions on the $\Omega_{0,ij}$ components are

$$\Omega_{0,5\bar{\alpha}} = \Omega_{0,5\alpha} = \Omega_{0,5\bar{5}} = \Omega_{0,\beta}{}^{\beta} = \Omega_{0,\beta_1\beta_2} = 0 \ , \tag{10.3}$$

and the traceless part of $\Omega_{0,\alpha\bar{\beta}}$ is not determined. The conditions on the $\Omega_{\bar{\alpha},ij}$ components are

$$\begin{aligned} \Omega_{\bar{\beta}_1,\bar{\beta}_2\bar{\beta}_3} &= 0 \ , \quad \Omega_{\bar{\alpha},\beta_1\beta_2} = -\Omega_{0,0[\beta_1}g_{\beta_2]\bar{\alpha}} \ , \quad \Omega_{\beta,\bar{\alpha}}{}^{\beta} = -\frac{3}{2}\Omega_{\bar{5},\bar{\alpha}5} = -\frac{3}{2}\Omega_{0,0\bar{\alpha}} \ , \\ \Omega_{\alpha,\beta}{}^{\beta} &= -\frac{1}{2}\Omega_{\bar{5},\alpha5} = -\frac{1}{2}\Omega_{0,0\alpha} \ , \end{aligned} \tag{10.4}$$

and the traceless part of $\Omega_{\alpha,\beta\bar{\gamma}}$ is not determined. In addition, we have

$$\Omega_{\bar{\beta}_1,\bar{\beta}_2\bar{5}} = \Omega_{\bar{\beta}_1,\bar{\beta}_25} = 0 \ , \quad \Omega_{(\bar{\alpha},\beta)\bar{5}} = \Omega_{(\bar{\alpha},\beta)5} = 0 \ , \quad \Omega_{\bar{\alpha},5\bar{5}} = 0 \ . \tag{10.5}$$

Finally, the conditions on the $\Omega_{\bar{5},ij}$ components are

$$\begin{aligned} &\Omega_{5,\beta}{}^{\beta} = \Omega_{\bar{5},\beta}{}^{\beta} = 0 \ , \quad \Omega_{5,\bar{\alpha}5} = \Omega_{\bar{5},\bar{\alpha}\bar{5}} = 0 \ , \quad \Omega_{5,\bar{\alpha}\bar{5}} = \Omega_{\bar{5},\bar{\alpha}5} \ , \\ &\Omega_{\bar{5},\bar{\alpha}5} = \Omega_{0,0\bar{\alpha}} \ , \quad \Omega_{\bar{5},5\bar{5}} = -\Omega_{5,5\bar{5}} = 0 \ , \quad \Omega_{5,\bar{\alpha}_1\bar{\alpha}_2} = \Omega_{\bar{5},\bar{\alpha}_1\bar{\alpha}_2} = 0 \ . \end{aligned} \tag{10.6}$$

The condition that remains to be examined is (6.29) or equivalently (6.36). This gives that

$$\Omega_{\bar{\alpha},5}{}^{\bar{\alpha}} = 0 \ . \tag{10.7}$$

The above equations, together with their complex conjugates, give full set of conditions that are required for a background to admit $N=4$ supersymmetry with Killing spinors given by $\eta_1 = f_1\eta^{SU(5)}$, $\eta_2 = f_2\theta^{SU(5)}$, $\eta_3 = f_3\eta^{SU(4)}$ and $\eta_4 = f_4\theta^{SU(4)}$.

Using the above conditions, the fluxes can be expressed in terms of the connection. For the electric part of the flux, we get

$$\begin{aligned} &G_{\alpha\beta\gamma} = G_{5\beta\gamma} = G_{\bar{5}\beta\gamma} = G_{\bar{\alpha}\beta\gamma} = 0 \ , \\ &G_{\bar{\alpha}5\gamma} = -2i\Omega_{\bar{\alpha},5\gamma} \ , \quad G_{\bar{5}5\alpha} = 3i\Omega_{0,0\alpha} \ . \end{aligned} \tag{10.8}$$

Similarly, for the magnetic part of the flux we find

$$\begin{aligned} F_{\alpha_1\alpha_2\alpha_3\alpha_4} = F_{5\alpha_1\alpha_2\alpha_2} = F_{\bar{\alpha}\beta_1\beta_2\beta_3} = F_{\bar{5}\beta_1\beta_2\beta_3} = 0 \ , \\ F_{\bar{\alpha}5\beta_1\beta_2} = F_{5\bar{5}\alpha}{}^{\alpha} = F_{\alpha\bar{\beta}\gamma}{}^{\gamma} = F_{\alpha\beta5\bar{5}} = 0 \ , \\ F_{\alpha\bar{\beta}5\bar{5}} = -2i\Omega_{0,\alpha\bar{\beta}} \ , \quad F_{\bar{\alpha}\bar{5}\beta_1\beta_2} = 0 \ . \end{aligned} \tag{10.9}$$

The components of the flux that do not appear in the above equations are not determined by the Killing spinor equations. Note that many of the components of the fluxes vanish as a consequence of the requirement of supersymmetry.

10.1 Geometry

We shall now investigate some aspects of the spacetime geometry that arises from the relations (10.1)-(10.7). We shall not elaborate on the description of the geometry because the properties of spacetime in this case are similar to that we have obtained for $N = 2$ and $N = 3$ backgrounds. The spacetime is equipped an $SU(4)$-structure. This can been seen from the forms associated with the spinors $\eta^{SU(5)}$, $\theta^{SU(5)}$, $\eta^{SU(4)}$ and $\theta^{SU(4)}$. In addition, the spacetime admits a timelike Killing vector field κ^f and the ten-dimensional space B transverse to the orbits of this vector field is a Hermitian manifold. Furthermore, B has two (real) holomorphic vector fields $\tilde{\kappa}^f = if^2(\partial_5 - \partial_{\bar{5}})$ and $\hat{\kappa}^f = f^2(\partial_5 + \partial_{\bar{5}})$. All three vector fields $\kappa^f, \tilde{\kappa}^f$ and $\hat{\kappa}^f$ commute.

The space $\hat{B}$ transverse to the orbits of all three vector fields is Hermitian with respect to the complex structure associated with the two-form $\omega^{SU(4)}$, see (4.24). In addition $\hat{B}$ admits an $SU(4)$-structure. We can adapt coordinates to all the above vector fields and write the metric as (9.9).

To summarize, we have seen that the $N = 4$ backgrounds with $SU(4)$ invariant spinors admit three commuting Killing vector fields κ^f, $\tilde{\kappa}^f$ and $\hat{\kappa}^f$. The first is timelike while the other two are spacelike. The space B transverse to the orbits of κ^f is a Hermitian manifold with an $SU(4)$-structure and two holomorphic Killing vector fields. The space $\hat{B}$ transverse to all three vector fields is also Hermitian with an $SU(4)$-structure. The $SU(4)$-structures on B and $\hat{B}$ are determined by the conditions (10.2)-(10.7), see also appendix B. Of course these conditions can be put into real form using the complex structure and the forms associated with these spinors. Note that the $SU(4)$-structures in the $N = 3$ and $N = 4$ backgrounds are different. Some of the components of the spin connection vanish in the latter but they do not vanish in the former.

11 Calabi-Yau compactifications with fluxes to one-dimension

One way to define the Calabi-Yau compactifications of M-theory with fluxes to one-dimension is to require[12] that the associated background is invariant under the one-dimensional Poincaré group and that the Killing spinors have stability subgroup $SU(5)$. In the absence of fluxes, it is clear that such backgrounds become the standard compactification of M-theory on ten-dimensional Calabi-Yau manifolds. According to this definition, these Calabi-Yau compactifications with fluxes are a special class of $N = 1$ and $N = 2$ supersymmetric backgrounds with $SU(5)$ invariant spinors. In the absence of fluxes the backgrounds preserve two supersymmetries. In the presence of fluxes, this is no longer the case. As we shall see the conditions that arise from the Killing spinor equations allow for backgrounds with only one supersymmetry.

The Poincaré group in one-dimension is $\mathbb{Z}_2 \ltimes \mathbb{R}$. In particular, the Lorentz group is $\mathbb{Z}_2$ and the non-trivial element acts on the one-dimensional Minkowski space as time inversion. In the context of the backgrounds we are investigating, the Lorentz group acts as $t \to -t$ on the coordinate t adapted to the Killing vector κ^f associated with the Killing spinor η_1. The subgroup $\mathbb{R}$ acts as translations on t.

Alternatively, one can require invariance of the background under the connected Poincaré group $\mathbb{R}$. As we have seen all $N = 1$ and $N = 2$ backgrounds with $SU(5)$ invariant spinors admit such a symmetry. Thus, we have already derived the supersymmetry conditions for such compactifications. In what follows, we shall focus on the $\mathbb{Z}_2 \ltimes \mathbb{R}$ case.

11.1 $N = 1$ compactifications

Requiring invariance under the $\mathbb{Z}_2$ Lorentz group, we find that the background can be written as

$$\begin{aligned} ds^2 &= -f^4 dt^2 + ds^2_{10} \\ \mathrm{F} &= F \ . \end{aligned} \tag{11.1}$$

In particular the off-diagonal term in the time component of the metric and the electric part of the flux vanish. Therefore $e^0 = f^2 dt$. Substituting this in the torsion free condition for the spin connection, we learn that $\Omega_{i,0j} = \Omega_{0,ij} = 0$. This in particular implies that

$$F_{\beta\bar{\alpha}\gamma}{}^{\gamma} = 0 \ . \tag{11.2}$$

Using (5.25) and (5.27), we find that the vanishing of the electric flux implies that

$$\Omega_{[\bar{\alpha}_1,\bar{\alpha}_2\bar{\alpha}_3]} = 0 \ , \qquad \Omega_{\bar{\alpha},\beta\gamma} = g_{\bar{\alpha}[\beta}\Omega_{0,0\gamma]} \ . \tag{11.3}$$

Substituting the first equation in (11.3) and $\Omega_{i,0j} = 0$ in (5.28), we get

$$F_{\bar{\alpha}\beta_1\beta_2\beta_3} = \frac{1}{2}\Omega_{\bar{\alpha},\bar{\gamma}_1\bar{\gamma}_2}\epsilon^{\bar{\gamma}_1\bar{\gamma}_2}{}_{\beta_1\beta_2\beta_3} \ . \tag{11.4}$$

[12] One may in addition require that the internal manifold is compact.

Taking the trace of the second equation in (11.3) and using (5.24), we find that

$$\Omega_{\bar{\alpha},}{}^{\bar{\alpha}}{}_{\gamma} = 2\Omega_{0,0\gamma} \ , \qquad \Omega_{\gamma,\beta}{}^{\beta} = \Omega_{0,0\gamma} \ . \tag{11.5}$$

Therefore, we have

$$\Omega_{\bar{\alpha},}{}^{\bar{\alpha}}{}_{\gamma} = 2\Omega_{\gamma,\beta}{}^{\beta} \ . \tag{11.6}$$

Substituting this into (5.19), we get

$$F_{\beta_1\beta_2\beta_3\beta_4} = -\frac{3}{2}\Omega_{0,0\bar{\alpha}}\epsilon^{\bar{\alpha}}{}_{\beta_1\beta_2\beta_3\beta_4} \ . \tag{11.7}$$

Therefore one concludes that the Killing spinor equations allow for M-theory Calabi-Yau compactifications with fluxes to one dimension with one supersymmetry. The non-vanishing fluxes are along the components $(4,0)+(0,4)$, $(1,3)+(3,1)$ and the traceless part of the $(2,2)$ component of the magnetic flux F. The $(4,0)+(0,4)$ and $(1,3)+(3,1)$ components of F are determined in terms of the ten-dimensional geometry of B. The conditions on geometry of B are given in (11.3) and in (11.6) . The scale factor f is also determined in terms of the geometry of B as

$$\Omega_{\gamma,\beta}{}^{\beta} + 2\partial_\gamma f = 0 \ . \tag{11.8}$$

11.2 $N = 2$ compactifications

The requirement of $N = 2$ supersymmetry imposes additional conditions on the geometry of spacetime for M-theory Calabi-Yau compactifications with fluxes. As in the $N = 1$ case above, the electric part of the flux and the off-diagonal part of the time component of the metric vanish as a consequence of the invariance of the background under the action of $\mathbb{Z}_2 \ltimes \mathbb{R}$. The latter condition again implies that $\Omega_{0,ij} = \Omega_{i,0j} = 0$. Using (6.27) and (6.28), we conclude that the $(1,3)+(3,1)$ components and the trace part of the (2,2) component of F vanish. Thus the only non-vanishing part of F is the traceless part of the (2,2) component. Since $G = 0$, (6.20) implies that $\Omega_{0,0\bar{\alpha}} = 0$ and in turn (6.24) gives that the scale factor f is constant. In addition $G = 0$ and (6.30) implies that

$$\Omega_{\bar{\alpha},\beta\gamma} = 0 \ . \tag{11.9}$$

From (6.22) we also get that

$$\Omega_{\alpha,\gamma}{}^{\gamma} = 0 \ . \tag{11.10}$$

So the geometry of the spacetime is simply $\mathbb{R} \times M_{CY}$, where M_{CY} is a ten-dimensional Calabi-Yau manifold. Therefore for M-theory $N = 2$ compactifications on Calabi-Yau manifolds with fluxes, the background is $\mathbb{R} \times M_{CY}$ and the only non-vanishing flux allowed is the traceless part of $F^{2,2}$. These are the conditions required by the Killing spinor equations.

The field equations in both the $N = 1$ and $N = 2$ cases will impose additional conditions on the geometry and flux F. The Einstein equations arise as integrability conditions of the Killing spinor equation and therefore the independent field equations are those of the flux [8]. The $N = 1$ case will be examined elsewhere [35], so we shall focus

on the $N = 2$ case here. The supergravity field equations for F are $d*\mathrm{F} + \frac{1}{2}\mathrm{F}\wedge\mathrm{F} = 0$. In the $N = 2$ case we are considering $\mathrm{F} = F$ and the only non-vanishing part is the traceless part of $F^{(2,2)}$. In particular $*F$ has an electric component and since the spacetime is $\mathbb{R} \times M_{CY}$, one concludes that $d * F = 0$ and $F \wedge F = 0$. Using the traceless condition $F \wedge F = 0$ implies that $|F|^2 = 0$ and therefore the field equations imply that the flux vanishes, i.e. $\mathrm{F} = 0$.

This result may change in M-theory because the field equations of F are modified by anomaly terms [28, 29]. However, it is likely that the inclusion of the anomaly terms into the theory changes the supersymmetry transformations in eleven dimensions leading to a new set of Killing spinor equations, see e.g. [30, 31, 32, 33]. In turn these should be re-investigated and the relation between geometry and fluxes may change. There has been recent progress in this for a class of M-theory compactifications [42, 43].

An alternative way to have non-vanishing fluxes in $N = 2$ compactifications is to allow F to have a non-vanishing electric component G and thus only require invariance only under the connected Poincaré group $\mathbb{R}$. An similar analysis has been done for M-theory compactifications with fluxes on eight-dimensional Calabi-Yau manifolds [34]. Such compactifications can also be reexamined using the conditions for supersymmetry we have derived for the $N = 4$ backgrounds with $SU(4)$ invariant spinors.

12 Concluding remarks

We have presented a method to directly solve the Killing spinor equations of eleven-dimensional supergravity. Our method is based on the gauge properties of the supercovariant connection and on a description of spinors in terms of forms. These have led to a better understanding of the Killing spinors of a supersymmetric M-theory background and to a simplification of the Killing spinor equations of eleven-dimensional supergravity. We have given the most general spinors that can arise in $N = 2$ backgrounds and we have solved the Killing spinor equations for two cases associated with $SU(5)$ and $SU(4)$ invariant spinors[13]. In the former case the geometry of spacetime is related to Hermitian ten-dimensional manifolds with an $SU(5)$-structure while in the latter case the geometry of spacetime is related to ten-dimensional almost Hermitian manifolds with an $SU(4)$-structure. In general, the G-structure of a spacetime is related to the stability subgroup of the Killing spinors in $Spin(1, 10)$. We have also presented two classes of $N = 3$ and $N = 4$ backgrounds with $SU(4)$ invariant spinors. In both cases the spacetime is related to Hermitian manifolds with an $SU(4)$-structure which admit holomorphic vector fields. Our method can be applied to ten-dimensional supergravities to extend the results of [6] to supersymmetric backgrounds with less than maximal supersymmetry. As an example of backgrounds with $N = 1$ and $N = 2$ supersymmetry, we have presented an application to M-theory Calabi-Yau compactifications with fluxes.

It would be of interest to use our formalism to express the Killing spinors of well-known M-theory backgrounds, e.g. M-branes [36, 37] and others, in terms of forms. This may lead to a better understanding of these solutions of the Killing spinor equations. One can use such a description in terms of forms to compute the spacetime forms associated

[13]The most general $N = 2$ case will be presented elsewhere [35].

with a pair of Killing spinors. In many cases such forms are associated with calibrations [21]. In the context of supergravity, they are not closed and give rise to generalized calibrations [38, 39]. The generalized calibrated submanifolds are the supersymmetric cycles in these backgrounds, i.e. they are the supersymmetric solutions of the M-brane worldvolume actions. It is worth pointing out that in [39] it was observed that the M2- and M5-brane backgrounds admit such generalized calibrations, see also [8]. These are most likely forms associated to the Killing spinors of these backgrounds.

As we have explained the emphasis of our method is in the description of Killing spinors. We expect that this will also assist in the physical interpretation of the various supersymmetric backgrounds that arise as the solutions of the Killing spinor equations. This is because most supersymmetric backgrounds with well-known physical interpretation, like e.g. M-branes, have been found by consideration of the expected isometries of the spacetime and the number of supersymmetries preserved [36, 37]. In addition, one of the criteria used to associate supergravity backgrounds to dual gauge theories is the number of supersymmetries that the former and the latter preserve. There may be a way to refine this supersymmetry criterion further by finding a direct interpretation of the stability group of the Killing spinors. We have already presented such an example in the context of M-theory Calabi-Yau compactifications with fluxes to one-dimension. Clearly the SU series of Killing spinors can be used to extend the definition of M-theory Calabi-Yau compactifications with fluxes to other Minkowski spaces. These paradigms can be extended further to other M-theory compactifications with fluxes. For example, one can *define* as M-theory G_2 compactifications with fluxes as those on backgrounds that exhibit the Poincaré symmetry of four-dimensional Minkowski space, (compact internal space) and admit Killing spinors that have stability subgroup G_2, see [40] for a review and [41] for the compactifications without fluxes. A similar characterization can be made for M-theory $Spin(7)$ and $Sp(2)$ compactifications with fluxes.

We have seen that the description of $N = 2$ backgrounds requires the investigation of a Gray-Hervella classification of geometric G-structures on a manifold. Some of them have been investigated already, like for example the standard $SU(5)$-structure in ten-dimensional Riemannian manifolds [23, 8, 24]. We have explored $SU(n-1)$-structures in $2n$-dimensional manifolds in appendix B. However it is clear from the description of $N = 2$ backgrounds that many other 'exotic' structures should be investigated. For example, we have seen that some $N = 2$ backgrounds exhibit an $SU(2)\times SU(2)$- and an $SU(2)\times SU(3)$-structure and that many other structures arise as stability subgroups of the Killing spinors.

It is clear that our method can be extended to analyze the Killing spinor equations of supergravity theories in lower dimensions. This is because the supercovariant derivatives of lower dimensional supergravity theories have a gauge group which includes an appropriate spin group and there is a description of spinors in terms of forms. It is expected that it will be straightforward to carry out our procedure for lower-dimensional supergravities with a small number of supersymmetries. This is because in such supergravities the space of spinors have a low dimension and the orbits of the spin groups have been investigated in [11]. This will provide an independent verification and a simplification of results that have already been obtained, see e.g. [44, 45], using other methods. It may also be possible to classify the supersymmetric solutions of lower-dimensional

supergravities with extended supersymmetry.

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Appendix A Orbits of special unitary groups

To understand the Killing spinors of eleven-dimensional supergravity, one has to find the orbits of $SU(5)$ and $SU(4)$ groups on $\Lambda^2(\mathbb{C}^5)$ and $\Lambda^2(\mathbb{C}^4)$, respectively. We shall argue here that the generic orbit of $SU(5)$ on $\Lambda^2(\mathbb{C}^5)$ is $SU(5)/SU(2)\times SU(2)$ and a representative complex two-form is $\sigma=\lambda_1 e^1\wedge e^2+\lambda_2 e^3\wedge e^4$, where $\lambda_1,\lambda_2\in\mathbb{R}$. While the generic orbit of $SU(4)$ in $\Lambda^2(\mathbb{C}^4)$ is $SU(4)/SU(2)\times SU(2)$ and a representative is $\sigma=\lambda_1 e^1\wedge e^2+\lambda_2 e^3\wedge e^4$, where $\lambda_1,\lambda_2\in\mathbb{C}$ obeying one real condition, i.e. $|\lambda_1|=1$. We shall produce two arguments for this. One is based on invariance and the other on a Lie algebra computation.

The independent $SU(5)$ invariant functions on $\Lambda^2(\mathbb{C}^5)$ are

$$\begin{aligned} I_1 &= ||\rho||^2=\delta^{a\bar a}\delta^{b\bar b}\rho_{ab}\bar\rho_{\bar a\bar b} \\ I_2 &= \delta^{b_1\bar a_2}\delta^{a_3\bar b_2}\delta^{b_3\bar a_4}\delta^{a_1\bar b_4}\rho_{a_1b_1}\rho_{\bar a_2\bar b_2}\rho_{a_3b_3}\rho_{\bar a_4\bar b_4} \end{aligned} \tag{A.1}$$

Both I_1, I_2 are real, so it is expected that the generic orbit has co-dimension two. To see that a representative of the generic orbit is $\sigma=\lambda_1 e^1\wedge e^2+\lambda_2 e^3\wedge e^4$, $\lambda_1,\lambda_2\in\mathbb{R}$, we shall demonstrate that $U^t\sigma U$, $U\in SU(5)$, is a generic two-form. In fact we shall show that $\rho=\sigma+i(H^t\sigma+\sigma H)$ is a generic two-form, where H is a Hermitian traceless matrix. In this way we will show that acting with $SU(5)$ transformations, one can span all the two-forms at at the linearized level of the action. Indeed we find that

$$\rho=\begin{pmatrix} 0 & \lambda_1(i+x_1+x_2) & \lambda_1 z_1-\lambda_2 y_3^* & \lambda_1 z_2+\lambda_2 y_2^* & \lambda_1 z_3 \\ -\lambda_1(i+x_1+x_2) & 0 & -\lambda_1 y_2-\lambda_2 z_2^* & -\lambda_1 y_3+\lambda_2 z_1^* & -\lambda_1 y_4 \\ -\lambda_1 z_1+\lambda_2 y_3^* & \lambda_1 y_2+\lambda_2 z_2^* & 0 & \lambda_2(i+x_4+x_3) & \lambda_2 v_1 \\ -\lambda_1 z_2-\lambda_2 y_2^* & -\lambda_2 z_1^*+\lambda_1 y_3 & -\lambda_2(i+x_3+x_4) & 0 & -\lambda_2 w_2 \\ -\lambda_1 z_3 & \lambda_1 y_4 & -\lambda_2 v_1 & \lambda_2 w_2 & 0 \end{pmatrix} \tag{A.2}$$

where

$$H=\begin{pmatrix} x_1 & y_1 & y_2 & y_3 & y_4 \\ y_1^* & x_2 & z_1 & z_2 & z_3 \\ y_2^* & z_1^* & x_3 & w_1 & w_2 \\ y_3^* & z_2^* & w_1^* & x_4 & v_1 \\ y_4^* & z_3^* & w_2^* & v_1^* & x_5 \end{pmatrix}, \qquad x_1+x_2+x_3+x_4+x_5=0\ ,\ \ x_1,\dots,x_5\in\mathbb{R}\ . \tag{A.3}$$

It is clear that if the lower diagonal entries of H are generic, then the lower diagonal entries of ρ are also generic apart perhaps from ρ_{21} and ρ_{43} which are determined from $x_1,\dots,x_5$ and λ_1,λ_2. From the conditions imposed on $x_1,\dots,x_5$, (x_1+x_2) is independent

from (x_3+x_4) and since λ_1 is independent from λ_2, ρ_{21} and ρ_{43} are also independent and ρ is a generic two form.

As a further confirmation, observe that the stability subgroup of σ is $SU(2)\times SU(2)$ and so the generic orbit is $SU(5)/SU(2)\times SU(2)$ which has dimension 18. On the other hand $\dim_{\mathbb{R}}\Lambda^2(\mathbb{C}^5)=20$. So $SU(5)/SU(2)\times SU(2)$ has codimension two as expected which is the number of independent parameters in σ.

It is clear that there are various special orbits for particular values of λ_1 and λ_2. To summarize, the various orbits are the following:

- $\lambda_1\neq\lambda_2\neq 0$: stability subgroup $SU(2)\times SU(2)$
- $\lambda_1=\lambda_2\neq 0$: stability subgroup $Sp(2)$
- $\lambda_1=0$ or $\lambda_2=0$: stability subgroup $SU(2)\times SU(3)$
- $\lambda_1=\lambda_2=0$: stability subgroup $SU(5)$.

The independent $SU(4)$ invariant functions on $\Lambda^2(\mathbb{C}^4)$ are

$$\begin{aligned} I_1 &= ||\rho||^2=\delta^{a\bar a}\delta^{b\bar b}\rho_{ab}\bar\rho_{\bar a\bar b} \\ I_2 &= \epsilon^{b_1b_2b_2b_4}\rho_{b_1b_2}\rho_{b_3b_4}\ , \end{aligned} \tag{A.4}$$

where ϵ is the holomorphic volume form. These are three independent functions and therefore it is expected that the generic orbit is of co-dimension three. A representative of this orbit is $\sigma=\lambda_1e^1\wedge e^2+\lambda_2e^3\wedge e^4$, $\lambda_1,\lambda_2\in\mathbb{C}$. As in the previous case, we shall show that $U^t\sigma U$, $U\in SU(4)$, spans $\Lambda^2(\mathbb{C}^4)$ by computing $\rho=\sigma+i(H^t\sigma+\sigma H)$. Indeed we find that

$$\rho=\begin{pmatrix} 0 & \lambda_1(i+x_1+x_2) & \lambda_1z_1-\lambda_2y_3^* & \lambda_1z_2+\lambda_2y_2^* \\ -\lambda_1(i+x_1+x_2) & 0 & -\lambda_1y_2-\lambda_2z_2^* & -\lambda_1y_3+\lambda_2z_1^* \\ -\lambda_1z_1+\lambda_2y_3^* & \lambda_1y_2+\lambda_2z_2^* & 0 & \lambda_2(i+x_4+x_3) \\ -\lambda_1z_2-\lambda_2y_2^* & -\lambda_2z_1^*+\lambda_1y_3 & -\lambda_2(i+x_3+x_4) & 0 \end{pmatrix} \tag{A.5}$$

where

$$H=\begin{pmatrix} x_1 & y_1 & y_2 & y_3 \\ y_1^* & x_2 & z_1 & z_2 \\ y_2^* & z_1^* & x_3 & w_1 \\ y_3^* & z_2^* & w_1^* & x_4 \end{pmatrix}\ , \qquad x_1+x_2+x_3+x_4=0\ ,\quad x_1,\dots,x_4\in\mathbb{R}\ . \tag{A.6}$$

It is clear that if the lower diagonal entries of H are generic, then the lower diagonal entries of ρ are also generic apart from ρ_{21} and ρ_{43}. These could be dependent because $x_1+x_2=-x_3-x_4$. However, we can allow for $\lambda_1,\lambda_2\in\mathbb{C}$. In such case, ρ_{21} and ρ_{43} are independent. They remain independent after imposing a condition on λ_1,λ_2, say $|\lambda_1|=1$. Thus we have shown that ρ is a generic two-form.

As a further confirmation, observe that the stability subgroup of σ is $SU(2)\times SU(2)$ and therefore the generic orbit is $SU(4)/SU(2)\times SU(2)$ which has dimension 9. On the other hand $\dim_{\mathbb{R}}\Lambda^2(\mathbb{C}^4)=12$. So $SU(4)/SU(2)\times SU(2)$ has codimension three as expected which is the number of independent parameters in σ.

There are various special orbits for particular values of λ_1 and λ_2. To summarize, the various orbits are the following:

- $\lambda_1 \neq \lambda_2 \neq 0$: stability subgroup $SU(2) \times SU(2)$
- $\lambda_1 = \lambda_2 \neq 0$: stability subgroup $Sp(2)$
- $\lambda_1 = 0$ or $\lambda_2 = 0$: stability subgroup $SU(2) \times SU(2)$
- $\lambda_1 = \lambda_2 = 0$: stability subgroup $SU(4)$.

Appendix B Geometric G-structures

B.1 Euclidean signature

Let X be a k-dimensional manifold equipped with a connection which takes values in the Lie algebra g and h be a Lie subalgebra $h \subset so(k)$ and $h \subset g$. One can classify the compatible h-structures in g as follows: Suppose that one can decompose $g = h \oplus h^\perp$, where we have assumed that there is a g-invariant inner product $<,>$ in g. Then the inequivalent h-structures are labeled by the irreducible representations of h in $\mathbb{R}^k \otimes h^\perp$, where $\mathbb{R}^k$ should be thought of as the vector representation of $so(n)$. This is equivalent to a decomposition of the so called intrinsic torsion K which is the difference of a g-connection and a compatible metric h connection, $K = \nabla^g - \nabla^h$. In all cases that we investigate below, there is a unique connection ∇^h such that K takes values in $\mathbb{R}^k \otimes h^\perp$ [22]. The vanishing of one or more components of K in the h-irreducible representation that arise in the decomposition of $\mathbb{R}^k \otimes h^\perp$ characterize the reduction of the g-structure to an h-structure. So if $\mathbb{R}^k \otimes h^\perp$ is decomposed in r h-irreducible representations, then there are 2^r inequivalent compatible h reductions of the g-structure. Then we say that the manifold admits an h-structure or an H-structure. In what follows we shall not be concerned with global topological issues which cannot be addressed from a local description of the supergravity solutions. Instead, we shall use the H-structures to characterize the geometry of supersymmetric backgrounds.

For applications to Riemannian manifolds, $g \subseteq so(k)$ and ∇^g is the Levi-Civita connection ∇. If the h-structure is characterized by the presence of invariant tensors, which we denote collectively with α, then the components of the intrinsic torsion can be represented by $\nabla\alpha$. In particular, this applies to the supersymmetric backgrounds which admit Killing spinors that have a non-trivial stability subgroup h. This is because the associated spacetime forms α constructed from the spinors are h invariant. However as we shall explain below, we can identify the components of the intrinsic torsion from those of the spin connection using the frame that comes naturally with the formalism we have developed in this paper. Of course this new way of identifying the components of the intrinsic torsion is equivalent to $\nabla\alpha$ and the two are related by a linear transformation.

For the Gray-Hervella classification of almost Hermitian manifolds one takes $g = so(2n)$, $so(k) = so(2n)$, $h = u(n)$, $n \geq 3$ and $\nabla^{so(2n)}$ to be the Levi-Civita connection. It is known that $\mathbb{R}^{2n} \otimes u(n)^\perp$ is decomposed into four irreducible representations under $u(n)$ and therefore there are sixteen almost Hermitian types of manifolds compatible with an $so(2n)$-structure. In the cases where we have an adapted $u(n)$ frame, as in the case of supersymmetric backgrounds that we are investigating, the intrinsic torsion can

be represented by the components of the $so(2n)$ spin-connection

$$\Omega_{\bar{\alpha},\beta\gamma}\ ,\qquad \Omega_{\bar{\alpha},\bar{\beta}\bar{\gamma}} \tag{B.1}$$

and their complex conjugates. To compare this with the standard definition of the intrinsic torsion as $\nabla\omega$, where ω is the Kähler form, we find that

$$\begin{aligned}\nabla_{\bar{\alpha}}\omega_{\beta\gamma} &= 2i\Omega_{\bar{\alpha},\beta\gamma}\\ \nabla_{\bar{\alpha}}\omega_{\bar{\beta}\bar{\gamma}} &= -2i\Omega_{\bar{\alpha},\bar{\beta}\bar{\gamma}}\ .\end{aligned} \tag{B.2}$$

The components $\Omega_{\bar{\alpha},\beta\bar{\gamma}}$ of the connection and their complex conjugates take values in $u(n)$. It suffices to decompose the (B.1) components of the intrinsic torsion under the four irreducible $u(n)$ representations in $\mathbb{R}^{2n}\otimes u(n)^{\perp}$ because the remaining components are not independent and they are related to the above by complex conjugation. The first component of the intrinsic torsion can be decomposed under $su(n)$ in a trace and a traceless part, i.e.

$$(\mathrm{w}_3)_{\gamma} = \Omega_{\bar{\beta},}{}^{\bar{\beta}}{}_{\gamma}\ ,\qquad (\mathrm{w}_4)_{\bar{\alpha}\beta\gamma} = \Omega_{\bar{\alpha},\beta\gamma} - \frac{2}{n-1}\Omega_{\bar{\delta},}{}^{\bar{\delta}}{}_{[\gamma}g_{\beta]\bar{\alpha}} \tag{B.3}$$

and the second component as

$$(\mathrm{w}_2)_{\bar{\alpha}\bar{\beta}\bar{\gamma}} = \Omega_{[\bar{\alpha},\bar{\beta}\bar{\gamma}]}\ ,\qquad (\mathrm{w}_1)_{\bar{\alpha}\bar{\beta}\bar{\gamma}} = \frac{2}{3}\Omega_{\bar{\alpha},\bar{\beta}\bar{\gamma}} - \frac{1}{3}\Omega_{\bar{\gamma},\bar{\alpha}\bar{\beta}} - \frac{1}{3}\Omega_{\bar{\beta},\bar{\gamma}\bar{\alpha}}\ . \tag{B.4}$$

The vanishing of one or more of the above four components of the intrinsic torsion characterize the sixteen classes of almost Hermitian manifolds. Using (B.2) one can relate the w classes to the $\mathcal{W}$ classes of the Gray-Hervella classification [22]. In particular, the w_3 class is related to the $\mathcal{W}_3$ class and similarly for the other classes.

In the above formalism it is straightforward to extend the classification of $u(n)$-structures to the $su(n)$-structures, $n\geq 3$ [23, 24]. In the latter case the independent components of the intrinsic torsion are

$$\Omega_{\bar{\alpha},\beta\gamma}\ ,\qquad \Omega_{\bar{\alpha},\bar{\beta}\bar{\gamma}}\ ,\qquad \Omega_{\bar{\alpha},\beta}{}^{\beta}\ . \tag{B.5}$$

The space $\mathbb{R}^{2n}\otimes su(n)^{\perp}$ decomposes into five irreducible representations under $su(n)$. Therefore there are thirty two inequivalent $su(n)$-structures compatible with an $so(2n)$-structure. The decomposition of the intrinsic torsion under $su(n)$ is

$$\begin{aligned}&(\mathrm{w}_3)_{\alpha} = \Omega_{\bar{\beta},}{}^{\bar{\beta}}{}_{\alpha}\ ,\qquad (\mathrm{w}_4)_{\bar{\alpha}\beta\gamma} = \Omega_{\bar{\alpha},\beta\gamma} - \frac{2}{n-1}\Omega_{\bar{\delta},}{}^{\bar{\delta}}{}_{[\gamma}g_{\beta]\bar{\alpha}}\ ,\qquad (\mathrm{w}_5)_{\bar{\alpha}} = \Omega_{\bar{\alpha},\beta}{}^{\beta}\ ,\\ &(\mathrm{w}_1)_{\bar{\alpha}\bar{\beta}\bar{\gamma}} = \Omega_{[\bar{\alpha},\bar{\beta}\bar{\gamma}]}\ ,\qquad (\mathrm{w}_2)_{\bar{\alpha}\bar{\beta}\bar{\gamma}} = \frac{2}{3}\Omega_{\bar{\alpha},\bar{\beta}\bar{\gamma}} - \frac{1}{3}\Omega_{\bar{\gamma},\bar{\alpha}\bar{\beta}} - \frac{1}{3}\Omega_{\bar{\beta},\bar{\gamma}\bar{\alpha}}\ .\end{aligned} \tag{B.6}$$

The vanishing of one or more of the above five components of the intrinsic torsion characterize the thirty two classes of almost Hermitian manifolds with an $su(n)$-structure.

Now let us turn to investigate the $u(n-1)$-structures of an $so(2n)$ manifold. For this we split the index $\alpha=(i,n)$, where $i=1,\dots,n-1$, $n\geq 4$. (The analysis below

works for $n \leq 4$ as well but some of the classes vanish identically.) The independent components of the intrinsic torsion in this case are

$$\begin{array}{llllllll} \Omega_{\bar{i},jk} \ , & \Omega_{\bar{i},\bar{j}\bar{k}} \ , & \Omega_{\bar{n},jk} \ , & \Omega_{\bar{n},\bar{j}\bar{k}} \ , & \Omega_{\bar{i},n\bar{n}} \ , & \Omega_{\bar{n},n\bar{n}} \ , & & \\ \Omega_{\bar{i},nj} \ , & \Omega_{\bar{i},\bar{n}j} \ , & \Omega_{\bar{i},n\bar{j}} \ , & \Omega_{\bar{i},\bar{n}\bar{j}} \ , & \Omega_{\bar{n},nj} \ , & \Omega_{\bar{n},\bar{n}j} \ , & \Omega_{\bar{n},n\bar{j}} \ , & \Omega_{\bar{n},\bar{n}\bar{j}} \ . \end{array} \tag{B.7}$$

The space $\mathbb{R}^{2n} \otimes u(n-1)^{\perp}$ decomposes into 20 irreducible representations under $u(n-1)$. Therefore there are 2^{20} inequivalent compatible $u(n-1)$-structures in an $so(2n)$ manifold. The intrinsic torsion decomposes under $u(n-1)$ as

$$\begin{array}{ll} (\mathrm{w}_3)_k = \Omega_{\bar{j},}{}^{\bar{j}}{}_k \ , & (\mathrm{w}_4)_{\bar{i}jk} = \Omega_{\bar{i},jk} - \dfrac{2}{(n-2)}\Omega_{\bar{m}}{}^{\bar{m}}{}_{[k}g_{j]\bar{i}} \ , \\ (\mathrm{w}_5)_{jk} = \Omega_{\bar{n},jk} \ , & (\mathrm{w}_6)_{\bar{j}\bar{k}} = \Omega_{\bar{n},\bar{j}\bar{k}} \ , \\ (\mathrm{w}_1)_{\bar{i}\bar{j}\bar{k}} = \Omega_{[\bar{i},\bar{j}\bar{k}]} \ , & (\mathrm{w}_2)_{\bar{i}\bar{j}\bar{k}} = \dfrac{2}{3}\Omega_{\bar{i},\bar{j}\bar{k}} - \dfrac{1}{3}\Omega_{\bar{k},\bar{i}\bar{j}} - \dfrac{1}{3}\Omega_{\bar{j},\bar{k}\bar{i}} \ , \\ (\mathrm{w}_7)_{\bar{i}} = \Omega_{\bar{i},n\bar{n}} \ , & \mathrm{w}_8 = \Omega_{\bar{n},n\bar{n}} \ , \end{array} \tag{B.8}$$

and another twelve. The first eight arise from taking traces and traceless parts of $\Omega_{\bar{i},nj}$ and $\Omega_{\bar{i},\bar{n}j}$, and symmetrizing and skew-symmetrizing $\Omega_{\bar{i},n\bar{j}}$ and $\Omega_{\bar{i},\bar{n}\bar{j}}$. The last four components in (B.7) are irreducible. The vanishing of one or more of the above components of the intrinsic torsion characterizes the 2^{20} inequivalent $u(n-1)$-structures of an $so(2n)$ manifold.

To investigate the $su(n-1)$-structures of an $so(2n)$ manifold, we again split the index $\alpha = (i, n)$, where $i = 1, \ldots, n-1$. The independent components of the intrinsic torsion in this case are

$$\begin{array}{llllllll} \Omega_{\bar{i},jk} \ , & \Omega_{\bar{i},\bar{j}\bar{k}} \ , & \Omega_{\bar{n},jk} \ , & \Omega_{\bar{n},\bar{j}\bar{k}} \ , & \Omega_{\bar{i},n\bar{n}} \ , & \Omega_{\bar{n},n\bar{n}} \ , & \Omega_{\bar{i},j}{}^{j} \ , & \Omega_{\bar{n},j}{}^{j} \ , \\ \Omega_{\bar{i},nj} \ , & \Omega_{\bar{i},\bar{n}j} \ , & \Omega_{\bar{i},n\bar{j}} \ , & \Omega_{\bar{i},\bar{n}\bar{j}} \ , & \Omega_{\bar{n},nj} \ , & \Omega_{\bar{n},\bar{n}j} \ , & \Omega_{\bar{n},n\bar{j}} \ , & \Omega_{\bar{n},\bar{n}\bar{j}} \ . \end{array} \tag{B.9}$$

The space $\mathbb{R}^{2n} \otimes su(n-1)^{\perp}$ decomposes into 22 irreducible representations under $su(n-1)$. Therefore there are 2^{22} inequivalent compatible $su(n-1)$-structures in an $so(2n)$ manifold. The intrinsic torsion decomposes under $su(n-1)$ as

$$\begin{array}{lll} (\mathrm{w}_3)_k = \Omega_{\bar{j},}{}^{\bar{j}}{}_k \ , & (\mathrm{w}_4)_{\bar{i}jk} = \Omega_{\bar{i},jk} - \dfrac{2}{(n-2)}\Omega_{\bar{m}}{}^{\bar{m}}{}_{[k}g_{j]\bar{i}} \ , & \\ (\mathrm{w}_5)_{jk} = \Omega_{\bar{n},jk} \ , & (\mathrm{w}_6)_{\bar{j}\bar{k}} = \Omega_{\bar{n},\bar{j}\bar{k}} \ , & (\mathrm{w}_9)_{\bar{i}} = \Omega_{\bar{i},j}{}^{j} \ , \\ (\mathrm{w}_1)_{\bar{i}\bar{j}\bar{k}} = \Omega_{[\bar{i},\bar{j}\bar{k}]} \ , & (\mathrm{w}_2)_{\bar{i}\bar{j}\bar{k}} = \dfrac{2}{3}\Omega_{\bar{i},\bar{j}\bar{k}} - \dfrac{1}{3}\Omega_{\bar{k},\bar{i}\bar{j}} - \dfrac{1}{3}\Omega_{\bar{j},\bar{k}\bar{i}} \ , & \\ (\mathrm{w}_7)_{\bar{i}} = \Omega_{\bar{i},n\bar{n}} \ , & \mathrm{w}_8 = \Omega_{\bar{n},n\bar{n}} \ , & (\mathrm{w}_{10}) = \Omega_{\bar{n},j}{}^{j} \ , \end{array} \tag{B.10}$$

and another twelve. The first eight arise from taking traces and traceless parts of $\Omega_{\bar{i},nj}$ and $\Omega_{\bar{i},\bar{n}j}$, and symmetrizing and skew-symmetrizing $\Omega_{\bar{i},n\bar{j}}$ and $\Omega_{\bar{i},\bar{n}\bar{j}}$. The last four components in (B.9) are irreducible. The vanishing of one or more of the above components of the intrinsic torsion characterizes the 2^{22} inequivalent $su(n-1)$-structures of an $so(2n)$ manifold.

B.2 Lorentzian signature

So far we have investigated g-structures for Euclidean signature manifolds. The analysis can be easily extended to the Lorentzian signature manifolds. We shall not examine Lorentzian case in detail since it is similar to the Euclidean case. Instead, we shall present an example of the $so(n)$-structures in an $so(n,1)$ manifold. This case is relevant in the context of $N=1$ supersymmetric backgrounds. The independent components of the intrinsic torsion in this case are

$$\Omega_{i,j0} \ , \quad \Omega_{0,0i} \ , \tag{B.11}$$

where we have split the frame index $A=(0,i)$, $i=1,\dots,n$. The space $\mathbb{R}^{n+1}\otimes so(n)^{\perp}$ decomposes under $so(n)$ into four irreducible representations. Therefore there are sixteen inequivalent $so(n)$-structures compatible with an $so(n,1)$-structure. The intrinsic torsion decomposes under $so(n)$ as

$$(w_1)_{ij}=\Omega_{[i,j]0} \ , \quad w_2=\Omega_{i,}{}^{i}{}_{0} \ , \quad (w_3)_{ij}=\Omega_{(i,j)0}-\frac{1}{n}g_{ij}\Omega_{k,}{}^{k}{}_{0} \ , \quad (w_4)_i=\Omega_{0,0i} \ . \tag{B.12}$$

The vanishing of one or more of the above components of the intrinsic torsion characterizes the sixteen inequivalent $so(n)$-structures of an $so(n,1)$ manifold. As we have seen the vanishing of the second and the third class in (B.12) is related to the existence of a time-like Killing vector on the Lorentzian manifold.

We can combine the results of this section with those we have presented for g-structures on Euclidean signature manifolds. In particular, we can find the $u(n)$-, $su(n)$-, $u(n-1)$- and $su(n-1)$-structures of an $so(2n,1)$ manifold. For example it is easy to see that there are 256 $u(n)$-structures in an $so(2n,1)$ manifold. The conditions for $N=1$ and $N=2$ supersymmetry, that we have derived in this paper, can be viewed as particular $su(5)$- and $su(4)$- structures in an $so(10,1)$ manifold. On the other hand, it is equivalent to express the various conditions arising from the Killing spinor equations in terms of the spacetime connection as we have done in the most part of the paper. The two ways of expressing the conditions of the spacetime geometry are related by a linear transformation.

Appendix C Spinors with stability subgroup $(Spin(7)\ltimes\mathbb{R}^8)\times\mathbb{R}$

To find a representative for the orbit of $Spin(1,10)$ with stability subgroup $(Spin(7)\ltimes\mathbb{R}^8)\times\mathbb{R}$, it is sufficient to find a spinor associated to a null vector. We first consider the spinor

$$ae_1+be_{2345} \ , \qquad a,b\in\mathbb{C} \ . \tag{C.1}$$

The Majorana condition implies that

$$a=\bar{b} \ . \tag{C.2}$$

Therefore, we construct two real spinors

$$e_1+e_{2345}$$

$$i(e_1 - e_{2345}) \ . \tag{C.3}$$

Using these, we write

$$\eta^{Spin(7)} \;=\; \frac{1}{2}(i(1 - e_{12345}) + e_1 + e_{2345}) \ . \tag{C.4}$$

Next we compute the associated vector to find

$$\begin{aligned} \kappa_0(\eta^{Spin(7)}, \eta^{Spin(7)}) &= B(\eta^{Spin(7)}, \Gamma_0 \eta^{Spin(7)}) = -1 \\ \kappa_1(\eta^{Spin(7)}, \eta^{Spin(7)}) &= B(\eta^{Spin(7)}, \Gamma_1 \eta^{Spin(7)}) = 1 \end{aligned} \tag{C.5}$$

with the rest of the components to vanish, i.e. $\kappa = -e^0 + e^1$. Clearly this is null and $\eta^{Spin(7)}$ is a representative of the orbit $\mathcal{O}_{Spin(7)}$.

Appendix D The solutions of the Killing spinor equations for $SU(4)$ invariant spinors

To solve the Killing spinor equations for the $SU(4)$ invariant spinor, we substitute the expression for the fluxes we have derived in the $N = 1$ case. Then we use the resulting equations to determine the remaining components of the fluxes and find the independent components of the spin connection. Throughout this calculation, we use the condition $\Omega_{i,0j} = \Omega_{0,ij}$ which arises from the Killing spinor equations for η_1 and (5.24) which expresses $\Omega_{0,0i}$ in terms of the geometry of the ten-dimensional manifold B.

First we investigate the solutions of the Killing spinor equations that involve a derivative along the frame time direction. Using (5.19) and (5.20), (7.3) and its complex conjugate imply that

$$\Omega_{0,0\bar{5}} = \Omega_{0,05} \tag{D.1}$$

and

$$-2\Omega_{0,05} + (\Omega_{5,5\bar{5}} - \Omega_{\bar{5},5\bar{5}}) + (\Omega_{5,\beta}{}^{\beta} - \Omega_{\bar{5},\beta}{}^{\beta}) = 0 \ . \tag{D.2}$$

Next we observe that using (5.23) and (5.27), (7.4) implies that

$$4\Omega_{0,\bar{\alpha}5} + i\Omega_{\beta_1,\beta_2\beta_3}\epsilon^{\beta_1\beta_2\beta_3}{}_{\bar{\alpha}} = 0 \ . \tag{D.3}$$

The equation (7.5) and its complex conjugate imply that

$$\partial_0 \log g = 0 \tag{D.4}$$

and

$$\Omega_{0,\beta}{}^{\beta} - \Omega_{0,5\bar{5}} + \frac{i}{12}F_{\alpha}{}^{\alpha}{}_{\beta}{}^{\beta} - \frac{i}{6}F_{\alpha}{}^{\alpha}{}_{5\bar{5}} = 0 \ . \tag{D.5}$$

The latter using (5.23) can be rewritten as

$$\Omega_{0,5\bar{5}} = \Omega_{0,\gamma}{}^{\gamma} \ . \tag{D.6}$$

Eliminating the fluxes in (7.6) using (5.28), we find

$$i\Omega_{5,\bar{\beta}_1\bar{\beta}_2} + i\Omega_{[\bar{\beta}_1,\bar{\beta}_2]\bar{5}} - \frac{1}{2}\Omega_{0,\gamma_1\gamma_2}\epsilon^{\gamma_1\gamma_2}{}_{\bar{\beta}_1\bar{\beta}_2} = 0 \ . \tag{D.7}$$

Similarly using (5.25) and (5.28), we get from (7.7) that

$$2\Omega_{\bar{5},\bar{5}\bar{\alpha}} + 2\Omega_{5,\bar{\alpha}\bar{5}} - 5\Omega_{0,0\bar{\alpha}} - 2\Omega_{\beta,\bar{\alpha}}{}^{\beta} = 0 \ . \tag{D.8}$$

Next we turn to investigate the condition that are associated with the Killing spinor equations involving the spatial derivative along the $\bar{\alpha}$ frame direction. It is easy to see, using (5.23) and (5.27), that (7.8) gives (D.3) and so (7.8) is not independent. Next, substituting (5.25) and (5.28) into (7.9), we find that

$$\Omega_{\bar{\alpha},\bar{\beta}5} - \Omega_{\bar{\alpha},\bar{\beta}\bar{5}} + \frac{1}{3}\Omega_{5,\bar{\alpha}\bar{\beta}} - \Omega_{\bar{5},\bar{\alpha}\bar{\beta}} - \frac{2}{3}\Omega_{[\bar{\alpha},\bar{\beta}]\bar{5}} - \frac{i}{3}\Omega_{0,\gamma_1\gamma_2}\epsilon^{\gamma_1\gamma_2}{}_{\bar{\alpha}\bar{\beta}} = 0 \ . \tag{D.9}$$

Eliminating the fluxes from (7.10) using (5.28) and (5.25), we get

$$\partial_{\bar{\alpha}} \log g + \frac{1}{2}\Omega_{\bar{\alpha},\gamma}{}^{\gamma} - \frac{1}{2}\Omega_{\bar{\alpha},5\bar{5}} - \frac{1}{6}\Omega_{\beta,\bar{\alpha}}{}^{\beta} + \frac{1}{6}\Omega_{5,\bar{\alpha}\bar{5}} + \frac{2}{3}\Omega_{\bar{5},\bar{5}\bar{\alpha}} - \frac{1}{6}\Omega_{0,0\bar{\alpha}} = 0 \ . \tag{D.10}$$

Similarly using (5.28) and (5.25), (7.11) gives

$$\Omega_{\bar{\alpha},\gamma_1\gamma_2}\epsilon^{\gamma_1\gamma_2}{}_{\bar{\beta}_1\bar{\beta}_2} + [\frac{1}{3}\Omega_{5,5\delta} - \frac{1}{3}\Omega_{\bar{\beta},\delta}{}^{\bar{\beta}} + \frac{1}{3}\Omega_{\bar{5},\delta 5} + \frac{1}{6}\Omega_{0,0\delta}]\epsilon^{\delta}{}_{\bar{\alpha}\bar{\beta}_1\bar{\beta}_2} = 0 \ . \tag{D.11}$$

Next we find using (5.28) and (5.25) that the condition (7.12) gives (D.7). As can be seen using (5.22) and (5.23), we cannot eliminate all the components of the a fluxes from (7.13) and we get

$$i\Omega_{\bar{\alpha},0\gamma} + \frac{1}{2}F_{\bar{\alpha}\gamma 5\bar{5}} - \frac{i}{3}\Omega_{0,\beta}{}^{\beta} g_{\bar{\alpha}\gamma} - \frac{2i}{3}\Omega_{0,5\bar{5}} g_{\bar{\alpha}\gamma} = 0 \ . \tag{D.12}$$

Similarly using (5.25) and (5.23), (7.14) gives

$$\frac{1}{4}F_{\bar{\alpha}\bar{5}\gamma_1\gamma_2} - \frac{1}{8}(\Omega_{\bar{\alpha},\bar{\beta}_1\bar{\beta}_2} + \Omega_{[\bar{\alpha},\bar{\beta}_1\bar{\beta}_2]})\epsilon^{\bar{\beta}_1\bar{\beta}_2}{}_{\gamma_1\gamma_2} + \frac{i}{3}g_{\bar{\alpha}[\gamma_1}\Omega_{\bar{5},0\gamma_2]} = 0 \ . \tag{D.13}$$

The condition (7.15) relates the partial derivative of g to the geometry of spacetime. Using (5.25), one finds that

$$\partial_{\bar{\alpha}} \log g - \frac{1}{2}\Omega_{\bar{\alpha},\gamma}{}^{\gamma} - \frac{1}{2}\Omega_{\gamma,\bar{\alpha}}{}^{\gamma} + \frac{1}{2}\Omega_{\bar{\alpha},5\bar{5}} + \frac{1}{2}\Omega_{5,\bar{\alpha}\bar{5}} - \frac{1}{2}\Omega_{0,0\bar{\alpha}} = 0 \ . \tag{D.14}$$

Eliminating the fluxes from (7.16) using (5.25) and (5.19), we find

$$\frac{1}{3}[-\frac{1}{2}\Omega_{0,05} - \frac{1}{2}\Omega_{0,0\bar{5}} - \Omega_{5,\gamma}{}^{\gamma} + \Omega_{\bar{5},\gamma}{}^{\gamma} - \Omega_{5,5\bar{5}} + \Omega_{\bar{5},5\bar{5}}]g_{\bar{\alpha}\beta} + \Omega_{\bar{\alpha},\beta\bar{5}} + \Omega_{\beta,\bar{\alpha}\bar{5}} = 0 \ . \tag{D.15}$$

Similarly using (5.28), we get from (7.17) that

$$4i\Omega_{\bar{\alpha},0\bar{5}} - \Omega_{\gamma_1,\gamma_2\gamma_3}\epsilon^{\gamma_1\gamma_2\gamma_3}{}_{\bar{\alpha}} = 0 \ . \tag{D.16}$$

Next we turn to investigate the conditions that are associated with the Killing spinor equations involving the spatial derivative along the $\bar{5}$ frame direction. As can be seen using (5.22) and (5.23), the condition (7.18) gives

$$2\Omega_{0,5\bar{5}} + \Omega_{0,\gamma}{}^{\gamma} = 0 \tag{D.17}$$

and together with (D.6) imply

$$\Omega_{0,5\bar{5}} = \Omega_{0,\gamma}{}^{\gamma} = 0 \ . \tag{D.18}$$

Next we use (5.25) and (5.28) to write (7.19) as

$$\Omega_{\bar{5},\bar{\beta}5} - \frac{2}{3}\Omega_{5,\bar{\beta}\bar{5}} + \frac{1}{3}\Omega_{\bar{5},\bar{5}\bar{\beta}} - \frac{1}{3}\Omega_{\alpha,\bar{\beta}}{}^{\alpha} - \frac{5}{6}\Omega_{0,0\bar{\beta}} = 0 \ . \tag{D.19}$$

Using (5.25), we can express the partial derivative of g, (7.20), as

$$\partial_{\bar{5}} \log g - \Omega_{\bar{5},5\bar{5}} - \frac{1}{2}\Omega_{0,0\bar{5}} = 0 \ . \tag{D.20}$$

Eliminating the fluxes from (7.21) using (5.25) and (5.28), we find that

$$\frac{1}{2}\Omega_{5,\gamma_1\gamma_2}\epsilon^{\gamma_1\gamma_2}{}_{\bar{\beta}_1\bar{\beta}_2} + \frac{5}{6}\Omega_{\gamma_1,\gamma_2 5}\epsilon^{\gamma_1\gamma_2}{}_{\bar{\beta}_1\bar{\beta}_2} + \frac{1}{3}\Omega_{\bar{5},\gamma_1\gamma_2}\epsilon^{\gamma_1\gamma_2}{}_{\bar{\beta}_1\bar{\beta}_2} - \frac{4i}{3}\Omega_{0,\bar{\beta}_1\bar{\beta}_2} = 0 \ . \tag{D.21}$$

As we shall see (D.21) is not an independent condition. Similarly using (5.28), we find that condition (7.22) gives (D.16) and so it is not independent. It is straightforward to see using (5.27) and (5.23) that (7.23) implies (D.3) and therefore it is not independent. Next we find using (5.27) that the condition (7.24) gives

$$\Omega_{5,\beta_1\beta_2} + \Omega_{[\beta_1,\beta_2]5} = 0 \ . \tag{D.22}$$

One can show using (5.19) and (5.27) that (7.25) implies

$$\partial_{\bar{5}} \log g - \frac{2}{3}(\Omega_{\bar{5},\gamma}{}^{\gamma} - \Omega_{5,\gamma}{}^{\gamma}) + \frac{1}{3}\Omega_{\bar{5},5\bar{5}} + \frac{2}{3}\Omega_{5,5\bar{5}} - \frac{1}{6}\Omega_{0,0\bar{5}} + \frac{1}{3}\Omega_{0,05} = 0 \ , \tag{D.23}$$

and similarly using (5.19) that (7.26) gives

$$\Omega_{0,0\alpha} + 2\Omega_{\alpha,5\bar{5}} + 2\Omega_{\alpha,\gamma}{}^{\gamma} + 2\Omega_{\bar{5},\alpha\bar{5}} = 0 \ . \tag{D.24}$$

This concludes the substitution of the fluxes in terms of the geometry in the conditions that arise from the Killing spinor equations for η_2.

The conditions that we have derived involving the connection can be interpreted as restrictions on the geometry of spacetime. These can be solved to find the independent components of the connection. For this we use (5.24) which expresses $\Omega_{0,0i}$ in terms of the geometry of B. In particular, the component $\Omega_{0,05}$ of the connection can be expressed in terms of the geometry of B. As a result (D.1) and (D.2) can be written as

$$(\Omega_{\bar{\beta}}{}^{\bar{\beta}}{}_5 - \Omega_{\beta}{}^{\beta}{}_{\bar{5}}) - (\Omega_{5,\beta}{}^{\beta} + \Omega_{\bar{5},\beta}{}^{\beta}) - (\Omega_{5,5\bar{5}} + \Omega_{\bar{5},5\bar{5}}) = 0 \tag{D.25}$$

and

$$-(\Omega_{\bar{\beta}}{}^{\bar{\beta}}{}_5 + \Omega_{\beta}{}^{\beta}{}_{\bar{5}}) + 2(\Omega_{5,\beta}{}^{\beta} - \Omega_{\bar{5},\beta}{}^{\beta}) + 2(\Omega_{5,5\bar{5}} - \Omega_{\bar{5},5\bar{5}}) = 0 \ , \tag{D.26}$$

respectively. Alternatively, we can use (5.29) to find that (D.1) implies that

$$\partial_5 f - \partial_{\bar{5}} f = 0 \ . \tag{D.27}$$

The condition (D.3) can be solved in terms of $\Omega_{0,\bar{\alpha}5}$ to find

$$\Omega_{0,\bar{\alpha}5} = -\frac{i}{4}\Omega_{\beta_1,\beta_2\beta_3}\epsilon^{\beta_1\beta_2\beta_3}{}_{\bar{\alpha}} \ , \tag{D.28}$$

which together with (D.16) implies

$$\Omega_{0,\bar{\alpha}5} = \Omega_{0,\bar{\alpha}\bar{5}} \ . \tag{D.29}$$

The condition (D.4) implies that g is independent of the frame time direction. The condition (D.7) can be solved to reveal that

$$\Omega_{0,\beta_1\beta_2} = \frac{i}{4}(\Omega_{5,\bar{\gamma}_1\bar{\gamma}_2} + \Omega_{\bar{\gamma}_1,\bar{\gamma}_2\bar{5}})\epsilon^{\bar{\gamma}_1\bar{\gamma}_2}{}_{\beta_1\beta_2} \ . \tag{D.30}$$

The condition (D.8) restricts the geometry of the space B as can be easily seen using (5.24). Substituting (D.30) into (D.9), we find

$$\Omega_{\bar{\alpha},\bar{\beta}5} - \Omega_{\bar{\alpha},\bar{\beta}\bar{5}} + \Omega_{5,\bar{\alpha}\bar{\beta}} - \Omega_{\bar{5},\bar{\alpha}\bar{\beta}} = 0 \ . \tag{D.31}$$

In particular, this gives

$$\Omega_{(\bar{\alpha},\bar{\beta})5} - \Omega_{(\bar{\alpha},\bar{\beta})\bar{5}} = 0 \tag{D.32}$$

and

$$\Omega_{[\bar{\alpha},\bar{\beta}]5} + \Omega_{5,\bar{\alpha}\bar{\beta}} = 0 \ , \tag{D.33}$$

where in the last step we have used (D.22). The condition (D.11) will be examined later. The condition (D.12) determines the flux $F_{\bar{\alpha}\gamma 5\bar{5}}$ in terms of the connection. The condition (D.13) expresses $F_{\bar{\alpha}\bar{5}\gamma_1\gamma_2}$ in terms of the connection and by taking the trace and comparing to (5.28) we find

$$F_{\gamma\bar{5}\delta}{}^{\delta} = \Omega_{0,\gamma\bar{5}} = 0 \ . \tag{D.34}$$

This in turn implies, using (D.28), that the totally anti-symmetric part of the connection vanishes, i.e. $\Omega_{[\beta_1,\beta_2\beta_3]} = 0$. By adding (D.10) and (D.14), and using (D.8), we get

$$\partial_{\bar{\alpha}} g = \partial_{\bar{\alpha}} f \ . \tag{D.35}$$

The difference between (D.10) and (D.14), after using (D.8), is

$$\Omega_{\bar{\alpha},\beta}{}^{\beta} - \Omega_{\bar{\alpha},5\bar{5}} + \Omega_{\bar{5},\bar{5}\bar{\alpha}} - \frac{1}{2}\Omega_{0,0\bar{\alpha}} = 0 \ . \tag{D.36}$$

The condition (D.15) together with (D.2) implies that

$$\Omega_{(\bar{\alpha},\beta),\bar{5}} = \frac{1}{2} g_{\bar{\alpha}\beta}\Omega_{0,0\bar{5}} \ . \tag{D.37}$$

By also taking (D.1) into account we find

$$\Omega_{(\bar{\alpha},\beta)5} = \Omega_{(\bar{\alpha},\beta)\bar{5}} \ . \tag{D.38}$$

Taking the trace of the two relations above yields

$$\Omega_{\bar\gamma,}{}^{\bar\gamma}{}_{\bar 5} + \Omega_{\gamma,}{}^{\gamma}{}_{\bar 5} = 4\Omega_{0,0\bar 5} \ , \tag{D.39}$$

$$\Omega_{\bar\gamma,}{}^{\bar\gamma}{}_{5} + \Omega_{\gamma,}{}^{\gamma}{}_{5} = \Omega_{\bar\gamma,}{}^{\bar\gamma}{}_{\bar 5} + \Omega_{\gamma,}{}^{\gamma}{}_{\bar 5} \ . \tag{D.40}$$

By adding (D.20) and (D.23), we find

$$\partial_{\bar 5} \log g + \frac{1}{3}(\Omega_{5,\gamma}{}^{\gamma} - \Omega_{\bar 5\gamma}{}^{\gamma}) + \frac{1}{3}(\Omega_{5,5\bar 5} - \Omega_{\bar 5,5\bar 5}) - \frac{1}{6}\Omega_{0,05} = 0 \ . \tag{D.41}$$

Together with its complex conjugate this equation gives

$$(\partial_{\bar 5} - \partial_5) g = 0 \ . \tag{D.42}$$

If we instead subtract (D.23) and (D.20), we get

$$-(\Omega_{\bar 5,\gamma}{}^{\gamma} - \Omega_{5,\gamma}{}^{\gamma}) + 2\Omega_{\bar 5,5\bar 5} + \Omega_{5,5\bar 5} + \Omega_{0,05} = 0 \ , \tag{D.43}$$

where we have also used (D.1). Taking the complex conjugate of this equation, we find

$$\Omega_{5,5\bar 5} + \Omega_{\bar 5,5\bar 5} = 0 \ , \tag{D.44}$$

and

$$-(\Omega_{\bar 5,\gamma}{}^{\gamma} - \Omega_{5,\gamma}{}^{\gamma}) + \Omega_{\bar 5,5\bar 5} + \Omega_{0,05} = 0 \ . \tag{D.45}$$

Comparing the above equation with (D.2), we find

$$\Omega_{0,05} = \Omega_{5,5\bar 5} \tag{D.46}$$

and

$$\Omega_{5,\gamma}{}^{\gamma} = \Omega_{\bar 5\gamma}{}^{\gamma} \ . \tag{D.47}$$

Substituting (D.46) into (D.20), we get

$$\partial_{\bar 5} g = \partial_{\bar 5} f \ , \tag{D.48}$$

where we have also used (5.29). Therefore (D.35) and (D.48) imply that

$$f = g \ . \tag{D.49}$$

The equation (D.21) is not independent. One can see this by using (D.22) and by comparing with (D.30).

Let us now turn to investigate (D.11). This can be written as

$$\Omega_{\bar\alpha,\gamma_1\gamma_2} + [\frac{1}{3}\Omega_{5,5[\gamma_1} - \frac{1}{3}\Omega_{\bar\beta,[\gamma_1}{}^{\bar\beta} - \frac{1}{3}\Omega_{\bar 5,5[\gamma_1} + \frac{1}{6}\Omega_{0,0[\gamma_1}]g_{\gamma_2]\bar\alpha} = 0 \ . \tag{D.50}$$

Taking the trace, we get

$$\Omega_{\bar\beta,\alpha}{}^{\bar\beta} + \Omega_{5,5\alpha} + \Omega_{\bar 5,\alpha 5} + \frac{1}{2}\Omega_{0,0\alpha} = 0 \ . \tag{D.51}$$

It remains to investigate the equations, (D.51), (D.24), (D.36), (D.19) and (D.8). By adding (D.24) and (D.36) we find

$$2\Omega_{\alpha,5\bar{5}} + \Omega_{\bar{5},\alpha\bar{5}} + \Omega_{5,5\alpha} = 0 \tag{D.52}$$

and by subtracting we get

$$2\Omega_{\alpha,\beta}{}^{\beta} + \Omega_{0,0\alpha} + \Omega_{\bar{5},\alpha\bar{5}} - \Omega_{5,5\alpha} = 0 \ . \tag{D.53}$$

Eliminating $\Omega_{\bar{\beta},\alpha}{}^{\bar{\beta}}$ and $\Omega_{\alpha,\beta}{}^{\beta}$ using (D.51) and (D.53) from the α-component of (5.24), we find

$$\Omega_{5,5\alpha} + \Omega_{\bar{5},\alpha\bar{5}} - 2\Omega_{\alpha,5\bar{5}} = 0 \ . \tag{D.54}$$

Comparing this with (D.52), we get

$$\Omega_{\alpha,5\bar{5}} = 0 \ , \qquad \Omega_{5,5\alpha} + \Omega_{\bar{5},\alpha\bar{5}} = 0 \ . \tag{D.55}$$

Eliminating $\Omega_{\bar{\beta},\alpha}{}^{\bar{\beta}}$ using (D.51) from (D.19) and (D.8), we get

$$\Omega_{5,\alpha\bar{5}} - \frac{1}{3}\Omega_{\bar{5},\alpha 5} + \frac{2}{3}\Omega_{5,5\alpha} - \frac{2}{3}\Omega_{0,0\alpha} = 0 \tag{D.56}$$

and

$$\Omega_{5,5\alpha} + \Omega_{\bar{5},\alpha 5} - \Omega_{0,0\alpha} = 0 \ . \tag{D.57}$$

The former using the latter becomes

$$\Omega_{5,\alpha\bar{5}} + \Omega_{5,5\alpha} - \Omega_{0,0\alpha} = 0 \ . \tag{D.58}$$

Subtracting (D.57) from (D.58), we find that

$$\Omega_{5,\alpha\bar{5}} = \Omega_{\bar{5},\alpha 5} \ . \tag{D.59}$$

Substituting (D.58) into (D.51), and using (D.59), we get

$$\Omega_{\bar{\beta},\alpha}{}^{\bar{\beta}} = -\frac{3}{2}\Omega_{0,0\alpha} \ . \tag{D.60}$$

In turn substituting (D.58) and (D.60) into (D.50), we find that

$$\Omega_{\bar{\alpha},\beta_1\beta_2} + \Omega_{0,0[\beta_1}g_{\beta_2]\bar{\alpha}} = 0 \ . \tag{D.61}$$

We also find from (D.53) that

$$2\Omega_{\alpha,\beta}{}^{\beta} + \Omega_{\bar{5},\alpha\bar{5}} + \Omega_{\bar{5},\alpha 5} = 0 \ . \tag{D.62}$$

This concludes our analysis. The final results are summarized in section (7.2).

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